AN INTRODUCTION TO

POLYTECHNIC MATHEMATICS-I

SEMESTER-I

AS PER NEW REVISED SYLLABUS

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CONTENTS

Cube root of unity are 1, $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$

Here one is real and other two are complex numbers which are conjugate to the each other.

1.1.15 Nature of imaginary cube root of unity :

The imaginary cube root of unity are $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$

Let $\alpha = \frac{-1+i\sqrt{3}}{2}$ and $\beta = \frac{-1-i\sqrt{3}}{2}$

Now $\alpha^2 = (\frac{-1+i\sqrt{3}}{2})^2 = \frac{1+3i^2-2i\sqrt{3}}{4} = \frac{2(-1-i\sqrt{3})}{4} = \frac{-1-i\sqrt{3}}{2} = \beta$

$\therefore \alpha^2 = \beta$

Again $\beta^2 = \frac{-1+i\sqrt{3}}{2} = \alpha$

$\therefore$ One of the imaginary cube root of unity is the square of the other imaginary cube root of unity. If ω is the imaginary cube root of unity then the other imaginary cube root of unity is ω^2.

$\therefore 1, \omega, \omega^2$ are the cube root of unity.

$\therefore \omega$ is the cube root of unity.

$\Rightarrow \omega^3 = 1$

$\Rightarrow \omega^3 - 1 = 0$

$\Rightarrow (\omega-1)(\omega^2+\omega+1) = 0$

$\Rightarrow \omega^2+\omega+1 = 0$ (since, $\omega - 1 \neq 0$ as $\omega \neq 1$).

1.1.16 DeMoivre's Theorem :

Statement- If n is an integer positive or negative then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

As we have if $z = r(\cos\theta + i\sin\theta)$, then $z^2 = r^2(\cos\theta + i\sin\theta)^2$ *

amp z^2 = amp $(z.z) = 2\theta$. And $|z^2| = |z||z| = r^2$

Hence $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$ **

From * and ** we have $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

So it can be extended to $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for integral values of n.

This theorem is extended to state:

If n is a fraction positive or negative, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

So this theorem can be applied in extracting roots of complex number.

Illustration:

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$(\cos\theta + i\sin\theta)^{-4} = \cos(-4\theta) + i\sin(-4\theta) = \cos 4\theta - i\sin 4\theta$

$(\cos 5\theta + i\sin 5\theta)^6 = \{(\cos\theta + i\sin\theta)^5\}^6 = \cos 30\theta + i\sin 30\theta$

Worked out examples.

Ex 1. (i) Reduce to a+ ib form

Soln : $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = i = 0 + i = a + ib$

where a = 0 and b = 1

(ii) [QP 2011] Find conjugate of $\frac{1}{3+5i}$

Soln: $\frac{1}{3+5i} = \frac{(3-5i)}{(3+5i)(3-5i)} = \frac{(3-5i)}{9-25i^2} = \frac{(3-5i)}{9+25} = \frac{3-5i}{34} = \frac{3}{34} - \frac{5}{34}i$

$\therefore$ conjugate of $\frac{1}{3+5i}$ = conjugate of $\frac{3}{34} - \frac{5}{34}i = \frac{3}{34} + \frac{5}{34}i$ Ans.

Ex 2. Simplify: (i) $(1-i)(1+\frac{1}{i})$

Soln: $(1-i)(1+\frac{1}{i}) = 1 - i + \frac{1}{i} - 1 = -i + \frac{i}{i^2} = -i - i = -2i$

(ii)[QP 2011] $\sqrt{-81} + \sqrt{-64}$

Soln: $\sqrt{-81} + \sqrt{-64} = 9i + 8i = 17i$

(iii)[QP 2011] $i^2 + \frac{1}{i^2}$

i.e if $z = a + ib$, where a and b are real, $i = \sqrt{-1}$, $a^2 + b^2 \neq 0$, then $\sqrt{a^2+b^2}$ (that is positive square root of a^2+b^2) is known as **Modulus** of $a + ib$ and is denoted by $|z|$ or mod(z). Thus **mod(z) = mod (a + ib) =** $+\sqrt{a^2+b^2}$

or $|z| = |a + ib| = +\sqrt{a^2+b^2}$

Again $\tan POE = \frac{PE}{OE} = \frac{y}{x}$, the amplitude of the complex number $x + iy$ may be written as $\tan^{-1}\frac{y}{x}$. If $z = x + iy$ then **amp z** $= \tan^{-1}(\frac{y}{x})$ or, **arg z** $= \tan^{-1}(\frac{y}{x})$

Let modulus of complex number is denoted by r i.e $OP = r$ then $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ which means $x = r\cos\theta$ and $y = r\sin\theta$.

$\Rightarrow x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

Thus any complex number can be expressed as $r(\cos\theta + i\sin\theta)$ which is the **polar** form.

1.1.11 Properties of modulus

(i) Modulus of a complex number and its conjugate complex no is same.

Let $z = a + ib$ then $\bar{z} = a - ib$

$|z| = |a + ib| = \sqrt{a^2+b^2} = \sqrt{a^2+(-b)^2} = |\bar{z}|$

(ii) Modulus of the product of two complex numbers is equal to the product of modulus of those complex numbers.

Let $z_1 = a + ib$, $z_2 = c + id$

Now $|z_1 z_2| = |(a + ib)(c + id)| = |(ac - bd) + i(bc + ad)|$

$= \sqrt{(ac-bd)^2 + (bc+ad)^2}$

$= \sqrt{a^2c^2 + b^2d^2 - 2abcd + b^2c^2 + a^2d^2 + 2abcd}$

$= \sqrt{(a^2+b^2)(c^2+d^2)} = |z_1||z_2|$

(iii) Modulus of the quotient of two complex number is equal to the quotient of modulus of those complex numbers.

Let $z_1 = a + ib$, $z_2 = c + ib$ Then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (Proof is not shown at present)

(iv) $z\bar{z} = |z|^2 = |\bar{z}|^2$

1.1.12 Properties of amplitude :

(i) amp $z = -$ amp $\bar{z}$

(ii) For z_1 and z_2 two complex numbers amp $(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2)$

(iii) For z_1 and z_2 two complex numbers amp $\frac{z_1}{z_2} = \text{amp}(z_1) - \text{amp}(z_2)$

1.1.13 Square root of a complex number :

Since the square of a complex number is a complex number so we can write,

$\sqrt{a+ib} = x + iy$

Squaring both side, $a + ib = (x + iy)^2 \Rightarrow a + ib = (x^2 - y^2) + 2ixy$

Equating real and imaginery part, we have

$x^2 - y^2 = a$ ....(i) and $2xy = b$

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2$

$\Rightarrow x^2 + y^2 = \sqrt{a^2+b^2}$ ....(ii)

(ii) + (i) $\Rightarrow x^2 = \frac{1}{2}(\sqrt{a^2+b^2} + a)$

(ii) – (i) $\Rightarrow y^2 = \frac{1}{2}(\sqrt{a^2+b^2} - a)$

Since product $xy = b/2$ hence xy and b will have same sign. Thus b will be + ve, when x and y will both + ve or x and y will be both – ve. similarly, b will be –ve, when x and y will be of opposite sign. So, if $\sqrt{a+ib} = x + iy$ then $\sqrt{a-ib} = x - iy$. For if $\sqrt{a+ib} = x + iy$ then squaring we have

$a + ib = (x^2 - y^2) + 2ixy$ so that $x^2 - y^2 = a$ and $2xy = b$

$a - ib = (x^2 - y^2) - 2ixy = (x - iy)^2$ Hence, $\sqrt{a-ib} = x - iy$

1.1.14 Cube root of unity : Let $\sqrt[3]{1} = x \Rightarrow x^3 = 1$

$\Rightarrow x^3 - 1 = 0 \Rightarrow x^3 - 1^3 = 0 \Rightarrow (x - 1)(x^2 + x + 1) = 0$

$\Rightarrow x = 1$ or $x^2 + x + 1 = 0$

Now, $x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

Numbers. Now, $z_1 + z_2 = (a_1 + i\,b_1) + (a_2 + i\,b_2) = (a_1 + a_2) + i(b_1 + b_2)$

$= M + iN$ where $M = (a_1 + a_2)$ and $N = b_1 + b_2$

Addition of Complex Numbers is also a Complex Number.

(ii) Subtraction of Complex Number: Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two Complex Numbers. Now, $z_1 - z_2 = (a_1 + i\,b_1) - (a_2 + i\,b_2) = (a_1 - a_2) + i\,(b_1 - b_2)$

$= M + iN$ where $M = (a_1 - a_2)$ and $N = (b_1 - b_2)$

Subtraction of two or more Complex Numbers is also a Complex Number.

(iii) Multiplication of Complex Number : Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two Complex Numbers. Now, $z_1 \cdot z_2 = (a_1 + i\,b_1)\,.\,(a_2 + i\,b_2)$

$= a_1\,a_2 + i\,a_2 b_1 + i\,a_1 b_2 + i^2 b_1 b_2$

$= (a_1 a_2 - b_1 b_2) + i\,(a_2 b_1 + a_1 b_2) = M + iN$

where $M = (a_1 a_2 - b_1 b_2)$ and $N = (a_2 b_1 + a_1\,b_2)$

Multiplication of two or more Complex Numbers is also a Complex Number.

(iv) Division of Complex Number : Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ be two

Complex Numbers. Now, $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \dfrac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$

$$= \frac{a_1 a_2 + b_1 b_2 + i(a_2 b_1 - a_1 b_2)}{a_2^{\,2} + b_2^{\,2}}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^{\,2} + b_2^{\,2}} + i\frac{a_2 b_1 - a_1 b_2}{a_2^{\,2} + b_2^{\,2}}$$

$= M + iN$

Division of two or more Complex Numbers is also a Complex Number.

1.1.6 Conjugate of Complex Number:

If $z = a + i\,b$ is a complex number then its conjugate complex number will be $\bar{z} = a - ib$. Conjugate complex number is denoted by $\bar{z}$.

1.1.7 Properties of Conjugate of Complex Number:

(i) Addition of a complex number and its conjugate complex number is a real number. Thus $z + \bar{z} = (a + i\,b) + (a - i\,b) = 2a$, which is a real number.

(ii) Subtraction of a complex number and its conjugate complex number is an imaginery number. Thus $z - \bar{z} = (a + i\,b) - (a - i\,b) = i2b$ which is an imaginary number.

(iii) $z = \bar{z} \Rightarrow z$ is purely real.

(iv) $z.\bar{z} = (a + i\,b)(a - ib) = a^2 + b^2$

1.1.8 Geometrical Representation of Complex Number:

The straight line XOX′ and YOY′ intersects at right angles at the point O. Let A is a point on XOX′ such that OA = 1 unit. Take the point E such that OE = x unit. E represents the number $+x$. If x is – ve then E lies on OX′. Now, with centre O draw an arc AB which intersects the imaginary axis at the point B. B represents the point $i\,(= \sqrt{-1})$. Now we take a point C on OX such that OC = y unit. Now draw a circle which intersects the imaginary axis at the point D and D′ Now the point D will represent the number iy and point D′ will represent the number $-iy$. The point P is taken on XOY whose co-ordinates are (x, y). The point P will represent the complex number $x + iy$. The point P′(co-ordinates$(x, -y)$) will represent the complex number $x - iy$. If Q $(-x, y)$ is a point on 2nd quadrant then Q will represent the complex number $-x + iy$. Similarly if Q′$(-x, -y)$ is a point on 3rd quadrant then Q′ will represent the complex number $-x - iy$. If $z = x + iy$ then the point P(x, y) will represent the complex number z. So any complex number lying in a plane can be represented by a point in the same plane.

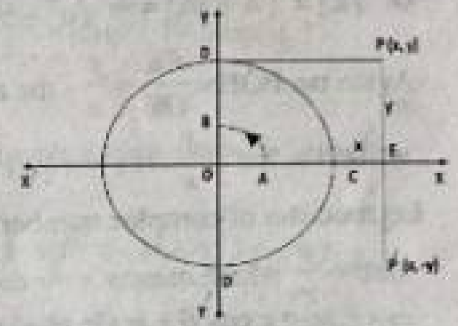

Fig1.1

1.1.9 Argand Diagram: The Fig 1.1 having the real and imaginary axes, in which plane of complex numbers are geometrically represented is called the Argand Diagram and the plane is also called complex plane or z-plane.

1.1.10 Modulus of Complex Number:

The complex number $x + iy$ can be represented in the Argand plane by the point P, then the linear distance of the point P from origin O is called the **modulus of the complex number** and the angle POE is called the **amplitude** (or argument) of the complex number.

Modulus of the complex number

$a + ib =$ OP $= \sqrt{a^2 + b^2}$

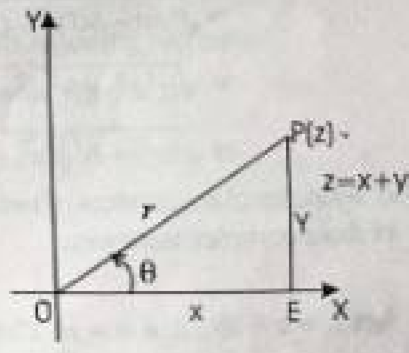

Fig 1.2

1.1 COMPLEX NUMBER

1.1.1 Introduction: We have discussed about Natural number, Real number, Rational number etc. Here we will introduce imaginary number. In the set of real number **R**, the equation $x^2 + 9 = 0$ has no solution. Because here $x^2 = -9$

$\therefore \quad x = \pm\sqrt{-9}$ which is impossible.

Since we know that square of a (positive or negative) number is always positive. To get solution of the equation, mathematician **Euler** introduce a symbol iota denoted by i, which is the first letter of the word imaginary. For the square root of -1, he introduce the property

$-1 = i^2$ [$i^2 = i.i = \sqrt{-1}\ \sqrt{-1} = (\sqrt{-1})^2 = -1$]

Now $\sqrt{-9}$ can be written as $\sqrt{-1}\sqrt{9} = \sqrt{i^2}\sqrt{9} = i\sqrt{9}$

1.1.2 Positive power of i

$i^4 = i^2 . i^2 = (-1)(-1) = 1$; $\quad i^{18} = (i^2)^9 = (-1)^9 = -1$

1.1.3 Definition of Complex Number:

If a and b are two real numbers then a number of the form $a + ib$ where $i = \sqrt{-1}$ is called a **Complex Number.** eg. $3 + 5i$, $4 - i7$, $-7 + 4i$ etc.

Note: 1. The Complex Number z can be written as $a + ib$.

2. Let $z = a + ib$ then 'a' is called real part of z and we write it as $\text{Re}(z) = a$. Similarly 'b' is called imaginary part of z and we write it as $\text{Im}(z) = b$

3. Every real number can be expressed as $r = r + i0$ (i.e. $a + ib$ form). So it is also a complex Number.

1.1.4 Equality of Complex Number: Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two Complex Numbers. Then, $z_1 = z_2$ if $a_1 = a_2$ and $b_1 = b_2$

1.1.5 Operation on Complex Number:

(i) Addition of Complex Number : Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two Complex

Soln: $i^2+\frac{1}{i^2}=-1-1=-2$

(iv)[QP 2011] i^{-106}

Soln: $i^{-106}=(i^2)^{-53}=(-1)^{-53}=-1$

(v)[QP2014] $\frac{2^n}{(1+i)^{2n}}+\frac{(1+i)^{2n}}{2^n}$

Soln: $\frac{2^n}{(1+i)^{2n}}+\frac{(1+i)^{2n}}{2^n}=\frac{2^n}{(2i)^n}+\frac{(2i)^n}{2^n}=(\frac{1}{i})^n+i^n=i^n\{(-1)^n+1^n\}$

If n is odd $\Rightarrow$ it is 0.

Ex 3.(i) [QP 2009, 2016] Express $\frac{2+3i}{2-i}$ with real denominators.

Soln: $\frac{2+3i}{2-i}=\frac{(2+3i)(2+i)}{(2-i)(2+i)}=\frac{1+8i}{4+1}=\frac{1}{5}(1+8i)$

(ii) [QP 2010] Transform $(2+i)(2-3i)(4-3i)$ into A+ iB form.

Soln: $(2+i)(2-3i)(4-3i)=(4+2i-6i-3i^2)(4-3i)=(4-4i+3)(4-3i)$
$=(7-4i)(4-3i)=28-16i-21i+12i^2$
$=28-37i-12=16-37i=16+(-37)i$
$=\text{A}+i\,\text{B}$

where A =16 and B = –37

(iii)[QP 2011] Express $\frac{2+3i}{5-4i}+\frac{2-3i}{5+4i}$ in the form $a+ib$.

Soln: $\frac{2+3i}{5-4i}+\frac{2-3i}{5+4i}=\frac{(2+3i)(5+4i)+(2-3i)(5-4i)}{(5-4i)(5+4i)}$

$=\frac{10+23i+12i^2+10-23i+12i^2}{25+16}=-\frac{4}{41}+i0$

(iv)[QP 2016] Express in the form A + iB: $\frac{(2+3i)^2}{2+i}$

Solution: i) $\frac{(2+3i)^2}{2+i}=\frac{-5+12i}{2+i}=\frac{(-5+12i)(2-i)}{(2+i)(2-i)}=\frac{1}{5}(2+29i)=\frac{2}{5}+i\frac{29}{5}$

Ex 4.(i)[QP 2009, 2013, 2015, 2017] Prove that $\sqrt{i}+\sqrt{-i}=\sqrt{2}$

Proof: LHS $=\sqrt{i}+\sqrt{-i}=\sqrt{(\sqrt{i}+\sqrt{-i})^2}=\sqrt{i+(-i)+2i(-i)}$
$=\sqrt{2}=$ RHS Proved.

(ii)[Q.P 2010] Prove that $\frac{1+2i+3i^2+4i^3}{5i^4+6i^5+7i^6+8i^7}=1$

Proof: LHS $=\frac{1+2i+3i^2+4i^3}{5i^4+6i^5+7i^6+8i^7}=\frac{1+2i-3-4i}{5+6i-7-8i}$
$=\frac{-2-2i}{-2-2i}=1=$ RHS Proved.

Ex 5.(i) Find out the Modulus of each of the following:

(a) $4+3i$ (b) $3-4i$ (c) $-i$ (d) $\frac{4}{1+i\sqrt{3}}$ (e)[2010] $\frac{3+4i}{12+5i}$

(f)[2010] $\frac{2+3i}{1-2i}$ (g) [2017] $\frac{12+5i}{24+7i}$

Solution: (a) mod$(4+3i)=\sqrt{4^2+3^2}=\sqrt{25}=5$

(b) mod$(3-4i)=\sqrt{3^2+4^2}=\sqrt{25}=5$

(c) $|-i|=|0+(-1)i|=\sqrt{0^2+(-1)^2}=1$

(d) $\left|\frac{4}{1+i\sqrt{3}}\right|=\left|\frac{4+i0}{1+i\sqrt{3}}\right|=\frac{|4+i0|}{|1+i\sqrt{3}|}=\frac{4}{2}=2$

(e) $\left|\frac{3+4i}{12+5i}\right|=\frac{|3+4i|}{|12+5i|}=\frac{\sqrt{3^2+4^2}}{\sqrt{12^2+5^2}}=\frac{5}{13}$

Exercise 1.1

1. Reduce to $a + ib$ form

(i) $\frac{a+ib}{c-id}$ (ii) $\frac{2+3i}{3-2i}$ (iii) $\frac{5}{1+i}$ (iv) $(i^{18}+(\frac{1}{i})^{25})^3$ (v) $(1-i)^3(2+i)$

Ans (i) $\frac{ac-bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}$ (ii) $0+i$ (iii) $\frac{5}{2}(1-i)$ (iv) $2-2i$ (v) $-2-6i$

2. Find the value of: (i) i^{52} (ii) i^{21} (iii) i^3+i^{-3} (iv) i^{27} (v) $\frac{1}{i^{44}}$

Ans (i) 1 (ii) i (iii) 0 (iv) $-i$ (v) 1

3. Find out the Modulus of each of the following.

(i) $\frac{1+i}{1-i}$ (ii) $\frac{3+4i}{12+5i}$ (iii) $\frac{1-2i}{2-i}$ (iv) $\frac{3+4i}{7+24i}$

Ans (i) 1 (ii) $\frac{5}{13}$ (iii) 1 (iv) $\frac{1}{5}$

4. Find out the amplitude of each of the following.

(i) $\frac{-4-4i}{3}$ (ii) $2+2i$ (iii) i (iv) $-2i$ (v) $\frac{1}{1+i\sqrt{3}}$

Ans (i) 225° (ii) 45° (iii) 90° (iv) -90° (v) -60°

5. Write in polar form: (i) $\frac{2+3i}{3-2i}$ (ii) $2+i2\sqrt{3}$ (iii) $(1+i)(1+2i)(1+3i)$

Ans (i) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ (ii) $4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$ (iii) $10(\cos\pi+i\sin\pi)$

6. Plot the points in the argand plane: (i) $-2-5i$ (ii) $3-i$ (iii) $3i$

7. Find the square root of: (i) $35+12i$ (ii) $21+20i$ (iii) $5-12i$ (iv) $7-24i$ (v) $15-8i$ (vi) i (vii) $72-54i$ (viii) $3+4i$ (ix) $9a^2-b^2-i6ab$ (x) $-1+\sqrt{24}\,i$

Ans: (i) $6+i$ (ii) $5+2i$ (iii) $3-2i$ (iv) $4-3i$ (v) $4-i$ (vi) $\frac{1}{\sqrt{2}}(1+i)$ (vii) $3(3-i)$ (viii) $2+i$ (ix) $3a-ib$ (x) $\sqrt{2}+\sqrt{3}\,i$

8. (i) If $x=-3+2i$ prove that $x^2+6x+13=0$

(ii) If $x=3+2i$ and $y=3-2i$ prove that $x^2-xy+y^2=-3$ and $x^3+y^3=-18$

(iii) If $x+iy=\frac{3}{\cos\theta+i\sin\theta}$ prove that $x^2+y^2=9$

(iv) If $x=2+i$ and $y=2-i$ prove that $x^3-y^3=-38i$, $x^3+y^3=124$

(v) If $x=4+3i$ and $y=4-3i$ prove that $x^2+xy+y^2=39$

9. If $z_1=2-i$, $z_1=1+i$ find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$ Ans $\sqrt{2}$

10. (i) Evaluate: (a) $(\omega^{12}-\frac{1}{\omega^{15}})^8$ (b) $(1+\omega^2+\omega^4+\omega^6)(1+\omega+\omega^3+\omega^5)$

(c) $\omega^5+\omega$ Ans (a) 0 (b) 1 (c) -1

(ii) Prove that (a) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)=1$

(b) $(2+\omega+\omega^2)^3+(-1+3\omega-\omega^2)^3=65$

(c) $(1-\omega+\omega^2)^5(1+\omega-\omega^2)^5=32$

(d) $(1+3\omega-\omega^2)^2=-12$

11. Show that $\frac{(\cos2\theta+i\sin2\theta)^3.(\cos3\theta-i\sin3\theta)^4}{(\cos3\theta+i\sin3\theta)^2.(\cos4\theta+i\sin4\theta)^{-3}}=1$

12. Show that $\frac{(\cos\theta+i\sin\theta)^8.(\cos2\theta+i\sin2\theta)^4}{(\cos3\theta+i\sin3\theta)^2.(\cos4\theta-i\sin4\theta)^{-2}}=1$

13. Find the value of i. $(-i)^{\frac{1}{6}}$ ii. $(1+i)^{\frac{1}{3}}$ iii. $1^{\frac{1}{3}}$

Ans- i. $\cos(4n+3)\frac{\pi}{12}+i\sin(4n+3)\frac{\pi}{12}$, $n=0, 1, 2, 3, 4, 5$

$$= \cos(n\pi + \frac{\pi}{4}) + i\sin(n\pi + \frac{\pi}{4}), n = 0, 1$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \quad (n=0), \quad \cos(\pi + \frac{\pi}{4}) + i\sin(\pi + \frac{\pi}{4}) \ (n=1)$$

$$= \frac{1}{\sqrt{2}}(1+i), \quad \frac{-1}{\sqrt{2}}(1+i)$$

Ex 13. Find the value of $\sqrt{1+i}$

Soln: $\sqrt{1+i} = 2^{\frac{1}{4}}\sqrt{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}} = 2^{\frac{1}{4}}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^{\frac{1}{2}}$

$$= 2^{\frac{1}{4}}\{\cos(2n\pi + \frac{\pi}{4}) + i\sin(2n\pi + \frac{\pi}{4})\}^{\frac{1}{2}}$$

$$= 2^{\frac{1}{4}}\{\cos(n\pi + \frac{\pi}{8}) + i\sin(n\pi + \frac{\pi}{8})\}, n = 0, 1$$

$$= 2^{\frac{1}{4}}\{\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\} \quad (n=0), \quad 2^{\frac{1}{4}}\{\cos(\pi + \frac{\pi}{8}) + i\sin(\pi + \frac{\pi}{8})\} \ (n=1)$$

$$= \pm 2^{\frac{1}{4}}(\frac{1}{2}\sqrt{2+\sqrt{2}} + i\frac{1}{2}\sqrt{2-\sqrt{2}}) = \pm 2^{\frac{1}{4}}(\frac{2^{\frac{1}{4}}}{2}\sqrt{\sqrt{2}+1} + i\frac{2^{\frac{1}{4}}}{2}\sqrt{\sqrt{2}-1})$$

$$= \pm\frac{1}{\sqrt{2}}(\sqrt{\sqrt{2}+1} + i\frac{1}{\sqrt{2}}\sqrt{\sqrt{2}-1})$$

Ex 14. Solve: $x^3 + 1 = 0$

Soln: $x = (-1)^{\frac{1}{3}} = (\cos\pi + i\sin\pi)^{\frac{1}{3}}$

$$= \{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}^{\frac{1}{3}}, n = 0, 1, 2$$

$$= \cos(\frac{2n\pi}{3} + \frac{\pi}{3}) + i\sin(\frac{2n\pi}{3} + \frac{\pi}{3})$$

Roots are $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, -1, \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

Or, $\frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, \frac{1}{2} - i\frac{\sqrt{3}}{2}$

Ex15. Solve: $x^4 + 1 = 0$

Solution: $x = (-1)^{\frac{1}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}}$

$$= \{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}^{\frac{1}{4}}, n = 0, 1, 2, 3$$

$$= \cos(\frac{2n\pi}{4} + \frac{\pi}{4}) + i\sin(\frac{2n\pi}{4} + \frac{\pi}{4})$$

Roots are $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}, \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4},$

$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$ Or $\pm(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}})$

$\Rightarrow x^2 = (7+25)/2 = 16 \quad \Rightarrow x = \pm 4$

$y^2 = (25-7)/2 = 9 \quad \Rightarrow y = \pm 3$

since $xy > 0$ so x and y are of same sign. Hence $\sqrt{7-24i} = \pm(4-3i)$

Ex 8. Prove that $\sqrt{-1-\sqrt{-1-\sqrt{-1-......}}} = \omega$ or ω^2

Solution: Let $\sqrt{-1-\sqrt{-1-\sqrt{-1-......}}} = x$

We have $x = \sqrt{-1-x} \quad \Rightarrow x^2 = -1-x$

$$\Rightarrow x^2+x+1=0 \quad \Rightarrow x = \frac{-1\pm i\sqrt{3}}{2} = \omega \text{ or } \omega^2$$

Ex9. If ω is imaginary cube root of unity prove that:

(i) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$

(ii) **[QP 2015]** $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$

(iii)**[QP 2009, 2013]** $(1-\omega+\omega^2)^2+(1+\omega-\omega^2)^2 = -4$

(iv)**[QP 2012]** $(1-\omega+\omega^2)^3+(1+\omega-\omega^2)^3 = -16$

(v)**[QP 2011, 2015]** $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5 = 32$

Solution: (i) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2), \omega^{10} = \omega,\ \omega^{11} = \omega^2$ since $\omega^3 = 1$

$= (2-\omega)^2(2-\omega^2)^2 = (4-4\omega+\omega^2)(4-4\omega^2+\omega), \qquad \omega^4 = \omega$

$= (1+\omega+\omega^2+3-5\omega)(1+\omega+\omega^2+3-5\omega^2)$ since $1+\omega+\omega^2 = 0$

$= (3-5\omega)(3-5\omega^2) = 9-15\omega-15\omega^2+25\omega^3$

$= 49-15\omega-15\omega^2-15 = 49-15(1+\omega+\omega^2) = 49-0 = 49$

(ii) $$\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega} = \frac{(2+\omega)(-\omega^2)+(1+2\omega)(-\omega^2)-(1+2\omega)(2+\omega)}{(1+2\omega)(2+\omega)(-\omega^2)}$$

$$= \frac{-5-5\omega-5\omega^2}{-4\omega^2-5} = \frac{-5(1+\omega+\omega^2)}{-4\omega^2-5} = 0$$

(iii) $(1-\omega+\omega^2)^2+(1+\omega-\omega^2)^2 = (1+\omega-2\omega+\omega^2)^2+(1+\omega+\omega^2-2\omega^2)^2$

$= (0-2\omega)^2+(0-2\omega^2)^2 = 4(\omega^2+\omega^4) = 4(\omega^2+\omega) = -4$

(iv) $(1-\omega+\omega^2)^3+(1+\omega-\omega^2)^3 = (1+\omega-2\omega+\omega^2)^3+(1+\omega+\omega^2-2\omega^2)^3$

$= (0-2\omega)^3+(0-2\omega^2)^3 = -8(\omega^3+\omega^6) = -8(1+1) = -16$

(v) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5 = (1+\omega-2\omega+\omega^2)^5+(1+\omega+\omega^2-2\omega^2)^5$

$= (0-2\omega)^5+(0-2\omega^2)^5 = -32(\omega^5+\omega^{10}) = -32(\omega^2+\omega) = 32$

Ex10. If $\sqrt[3]{a+ib} = x+iy$ prove that

(i) $\sqrt[3]{a-ib} = x-iy$ **(ii)** $\frac{a}{x}+\frac{b}{y} = 4(x^2-y^2)$

Solution: (i) $a+ib = (x+iy)^3 = x^3+i^3y^3+3i^2y^2x+3iyx^2$

$= (x^3-3xy^2)+i(3x^2y-y^3)$

$\Rightarrow a-ib = (x^3-3xy^2)-i(3x^2y-y^3) = x^3+iy^3-3y^2x-3iyx^2$

$= x^3-(iy)^3+3x(iy)^2-3x^2(iy) = (x-iy)^3$

(ii) From above example $a = x^3-3xy^2 \quad b = 3x^2y-y^3$

$$\frac{a}{x}+\frac{b}{y} = \frac{x^3-3xy^2}{x}+\frac{3x^2y-y^3}{y} = (x^2-3y^2)+(3x^2-y^2) = 4(x^2-y^2)$$

Ex11. Find the value of $(2+2i)^2$.

Soln: $(2+2i)^2 = 2^2(1+i)^2 = 4\{\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})\}^2 = 8(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^2$

$= 8(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}) = 8i$

Ex12. Find the value of $\sqrt{i}$

Soln: $\sqrt{i} = \sqrt{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}}$

$= (\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})^{\frac{1}{2}} = \{\cos(2n\pi+\frac{\pi}{2})+i\sin(2n\pi+\frac{\pi}{2})\}^{\frac{1}{2}}$

(f) mod $\frac{2+3i}{1-2i}=\frac{|2+3i|}{|1-2i|}=\frac{\sqrt{2^2+3^2}}{\sqrt{1^2+2^2}}=\sqrt{\frac{13}{5}}$

(g) $\left|\frac{12+5i}{24+7i}\right|=\frac{|12+5i|}{|24+7i|}=\frac{13}{25}$

(ii) Find the modulus and argument of the complex number

(a)[QP 2009] $\sqrt{3}+i$ **(b)** $\frac{1+2i}{1-3i}$

Solution: (a) $|\sqrt{3}+i|=\sqrt{3+1^2}=2$

$\arg(\sqrt{3}+i)=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$

(b) $\left|\frac{1+2i}{1-3i}\right|=\frac{|1+2i|}{|1-3i|}=\frac{\sqrt{1^2+2^2}}{\sqrt{1^2+3^2}}=\frac{1}{\sqrt{2}}$

$\arg(\frac{1+2i}{1-3i})=\arg(1+2i)-\arg(1-3i)=\tan^{-1}2-\tan^{-1}(-3)$

$=\tan^{-1}\frac{2-(-3)}{1+2\times(-3)}=\tan^{-1}\frac{1}{-1}=\frac{3\pi}{4}$

(iii) Convert the following to the polar form: **(a)** $\frac{1+7i}{(2-i)^2}$ **(b)** $\frac{1+3i}{1-2i}$

Solution: (a) $\frac{1+7i}{(2-i)^2}=\frac{1+7i}{3-4i}=\frac{(1+7i)(3+4i)}{(3-4i)(3+4i)}=\frac{3-28+25i}{25}=-1+i$

Now $\left|\frac{1+7i}{(2-i)^2}\right|=|-1+i|=\sqrt{2}$

$\arg(\frac{1+7i}{(2-i)^2})=\arg(-1+i)=\tan^{-1}\frac{1}{-1}=\frac{3\pi}{4}$

$\therefore$ polar form is $\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

(b) $\frac{1+3i}{1-2i}=\frac{(1+3i)(1+2i)}{(1-2i)(1+2i)}=\frac{1-6i+5i}{1+4}=-1+i$

Now $\left|\frac{1+3i}{1-2i}\right|=|-1+i|=\sqrt{2}$

$\arg(\frac{1+3i}{1-2i})=\arg(-1+i)=\tan^{-1}\frac{1}{-1}=\frac{3\pi}{4}$

$\therefore$ polar form is $\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

Ex6.(i) [QP 2011] If $x=1+i$ find the value of x^2-2x+2

Soln: $x^2-2x+2=x^2-2x+1+1=(x-1)^2+1=(1+i-1)^2+1=i^2+1=0$

(ii) If $x=3+2i$ and $y=3-2i$ find the value of x^2+xy+y^2

Soln: $x^2+xy+y^2=(x^2+y^2)+xy=(3+2i)^2+(3-2i)^2+(3+2i)(3-2i)$

$=2\{3^2+(2i)^2\}+(9+4)=10+13=23$

(iii) [QP 2014] If $z=a+ib$ and $|z-2|=|2z-1|$ prove that $a^2+b^2=1$

Soln: $|z-2|^2=|a-2+ib|^2=(a-2)^2+b^2=a^2+b^2-4a+4$

$|2z-1|^2=|(2a-1)+i2b|^2=(2a-1)^2+4b^2=4(a^2+b^2)-4a+1$

$\therefore a^2+b^2-4a+4=4(a^2+b^2)-4a+1$

$\Rightarrow 3(a^2+b^2)=3 \quad a^2+b^2=1$

Ex 7. Find the square root: **(i)** $3+4i$ **(ii)** $7-24i$

Soln: **(i)** Let $\sqrt{3+4i}=x+iy \Rightarrow x^2-y^2=3,\ 2xy=4,\ x^2+y^2=\sqrt{3^2+4^2}=5$

$\Rightarrow x^2=(3+5)/2=4 \quad x=\pm 2. \qquad y^2=(5-3)/2=1 \quad \Rightarrow y=\pm 1$

since $xy>0$ so x and y are of same sign. Hence $\sqrt{3+4i}=\pm(2+i)$

(ii) Let $\sqrt{7-24i}=x-iy \Rightarrow x^2-y^2=7,\ 2xy=24,\ x^2+y^2=\sqrt{7^2+24^2}=25$

$x=1 \Rightarrow 3=-6B \Rightarrow B=-1/2$
$x=2 \Rightarrow 7=12D \Rightarrow D=7/12$

Hence $\dfrac{x^2+x+1}{(x^2-1)(x^2-4)}=\dfrac{1}{6(x+1)}-\dfrac{1}{2(x-1)}-\dfrac{1}{4(x+2)}+\dfrac{7}{12(x-2)}$

Ex 7. $\dfrac{x^3+1}{x^3-x}$

$=\dfrac{x^3-x+x+1}{x^3-x}=1+\dfrac{x+1}{x(x-1)(x+1)}=1+\dfrac{1}{x(x-1)}$

We write $\dfrac{1}{x(x-1)}=\dfrac{A}{x}+\dfrac{B}{x-1}=\dfrac{A(x-1)+Bx}{x(x-1)}$

$\Rightarrow 1=A(x-1)+Bx$
$x=0 \Rightarrow 1=-A \Rightarrow A=-1$
$x=1 \Rightarrow 1=B$

Hence $\dfrac{x^3+1}{x^3-x}=1+\dfrac{1}{x(x-1)}=1+\dfrac{A}{x}+\dfrac{B}{x-1}=1-\dfrac{1}{x}+\dfrac{1}{x-1}$

Ex 8. $\dfrac{x^2}{x^4-1}$

$=\dfrac{x^2}{(x+1)(x-1)(x^2+1)}$

We write $\dfrac{x^2}{(x+1)(x-1)(x^2+1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{Cx+D}{x^2+1}$

$=\dfrac{A(x-1)(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x^2-1)}{(x+1)(x-1)(x^2+1)}$

$\Rightarrow x^2=A(x-1)(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x^2-1)$
$x=-1 \Rightarrow 1=-4A \Rightarrow A=-1/4$
$x=1 \Rightarrow 1=4B \Rightarrow B=1/4$

Equating the coefficients of x^3 we have $0=A+B+C=-1/4+1/4+C \Rightarrow C=0$
Equating the constant terms we have $0=-A+B-D=1/4+1/4-D \Rightarrow D=1/2$

Hence $\dfrac{x^2}{x^4-1}=\dfrac{x^2}{(x+1)(x-1)(x^2+1)}=-\dfrac{1}{4(x+1)}+\dfrac{1}{4(x-1)}+\dfrac{1}{2(x^2+1)}$

Ex 9. $\dfrac{1}{x(x^3+1)}$

$=\dfrac{1}{x(x+1)(x^2-x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{Cx+D}{x^2-x+1}$

$=\dfrac{A(x+1)(x^2-x+1)+Bx(x^2-x+1)+(Cx+D)x(x+1)}{x(x+1)(x^2-x+1)}$

$\Rightarrow 1=A(x+1)(x^2-x+1)+Bx(x^2-x+1)+(Cx+D)(x^2+x)$
$x=-1 \Rightarrow 1=-3B \Rightarrow B=-1/3$
$x=0 \Rightarrow 1=A$

Equating the coefficients of x^3 we have $0=A+B+C=1-1/3+C \Rightarrow C=-2/3$
Equating the coefficients of x we have $0=B+D=-1/3+D \Rightarrow D=1/3$

Hence $\dfrac{1}{x(x^3+1)}=\dfrac{1}{x}-\dfrac{1}{3(x+1)}-\dfrac{2x-1}{3(x^2-x+1)}$

Ex 10. $\dfrac{x^4}{(x^2-1)(x+2)}$

$=\dfrac{x^4}{x^3+2x^2-x-2}=\dfrac{(x-2)(x^3+2x^2-x-2)+(5x^2-4)}{x^3+2x^2-x-2}$

$=x-2+\dfrac{5x^2-4}{x^3+2x^2-x-2}=x-2+\dfrac{5x^2-4}{(x+1)(x-1)(x+2)}$

We write $\dfrac{5x^2-4}{(x+1)(x-1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{x+2}$

Ex 3. $\dfrac{x^2}{(x+1)(x+2)^2}$

$$=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}=\frac{A(x+2)^2+B(x+1)(x+2)+C(x+1)}{(x+1)(x+2)^2}$$

$\Rightarrow x^2=A(x+2)^2+B(x+1)(x+2)+C(x+1)$

$x=-1 \quad \Rightarrow 1=A$

$x=-2 \quad \Rightarrow 4=-C \Rightarrow C=-4$

Equating the coefficients of x^2 we have $1=A+B=1+B \Rightarrow B=0$

Hence $\dfrac{x^2}{(x+1)(x+2)^2}=\dfrac{1}{x+1}-\dfrac{4}{(x+2)^2}$

Ex 4. $\dfrac{2x}{(x-1)^3(x+1)}$

$$=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$$

$$=\frac{A(x-1)^3+B(x+1)(x-1)^2+C(x+1)(x-1)+D(x+1)}{(x+1)(x-1)^3}$$

$\Rightarrow 2x=A(x-1)^3+B(x+1)(x-1)^2+C(x+1)(x-1)+D(x+1)$

$x=1 \quad \Rightarrow 2=2D \quad \Rightarrow D=1$

$x=-1 \quad \Rightarrow -2=-8A \quad \Rightarrow A=1/4$

Equating the coefficients of x^3 we have

$0=A+B=1/4+B \quad \Rightarrow B=-1/4$

Equating the coefficients of x^2 we have

$0=-3A-B+C=-3/4+1/4+C \quad \Rightarrow C=1/2$

Hence $\dfrac{2x}{(x-1)^3(x+1)}=\dfrac{1}{4(x+1)}-\dfrac{1}{4(x-1)}+\dfrac{1}{2(x-1)^2}+\dfrac{1}{(x-1)^3}$

Ex 5. $\dfrac{(x^2+1)^2}{x^4+x^2+1}$

$$=\frac{x^4+x^2+1+x^2}{x^4+x^2+1}=1+\frac{x^2}{x^4+x^2+1}=1+\frac{x^2}{(x^2+1)^2-x^2}$$

$$=1+\frac{x^2}{(x^2+1+x)(x^2+1-x)}$$

We write $\dfrac{x^2}{(x^2+1+x)(x^2+1-x)}=\dfrac{Ax+B}{x^2+x+1}+\dfrac{Cx+D}{x^2-x+1}$

$\Rightarrow x^2=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$

Equating the coefficients of x^3 we have $0=A+C$ I

Equating the coefficients of x^2 we have $1=-A+B+C+D$ II

Equating the coefficients of x we have $0=A-B+C+D$ III

Equating the constant terms we have $0=B+D$ IV

From II and IV $\quad 1=-A+C$ V

From I and V $\quad C=1/2, \quad A=-1/2$

From I and III $\quad B=D$ and from IV $B=D=0$

Hence $\dfrac{(x^2+1)^2}{x^4+x^2+1}=1+\dfrac{x^2}{(x^2+1+x)(x^2+1-x)}$

$$=1-\frac{x}{2(x^2+x+1)}+\frac{x}{2(x^2-x+1)}$$

Ex 6. $\dfrac{x^2+x+1}{(x^2-1)(x^2-4)}$

$$=\frac{x^2+x+1}{(x-1)(x+1)(x+2)(x-2)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{x-2}$$

$$=\frac{A(x-1)(x^2-4)+B(x+1)(x^2-4)+C(x-2)(x^2-1)+D(x+2)(x^2-1)}{(x+1)(x-1)(x+2)(x-2)}$$

$\Rightarrow x^2+x+1$

$=A(x-1)(x^2-4)+B(x+1)(x^2-4)+C(x-2)(x^2-1)+D(x+2)(x^2-1)$

$x=-1 \quad \Rightarrow 1=6A \quad \Rightarrow A=1/6$

$x=-2 \quad \Rightarrow 3=-12C \quad \Rightarrow C=-1/4$

Case I- Factors of $g(x)$ are non-repeated linear factors.

$$g(x) = (ax+b)(cx+d)(ex+f).......$$

Then $$\frac{f(x)}{g(x)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f} + ...$$

Illustration: $$\frac{2x-3}{x^2+6x+8} = \frac{2x-3}{(x+4)(x+2)} = \frac{A}{x+4} + \frac{B}{x+2}$$

Case II- Factors of $g(x)$ are repeated linear factors.

$$g(x) = (ax+b)(cx+d)^n$$

Then $$\frac{f(x)}{g(x)} = \frac{A}{ax+b} + \frac{B_1}{cx+d} + \frac{B_2}{(cx+d)^2} + ... + \frac{B_n}{(cx+f)^n}$$

Illustration: $$\frac{x^2-2}{(x-3)(x+1)^3} = \frac{A}{(x-3)} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2} + \frac{B_3}{(x+1)^3}$$

Case III- Factors of $g(x)$ are non-repeated linear and quadratic factors.

$$g(x) = (ax+b)(cx^2+dx+e)$$

Then $$\frac{f(x)}{g(x)} = \frac{A}{ax+b} + \frac{Cx+D}{cx^2+dx+e}$$

Illustration: $$\frac{3}{(x-3)(6x^2+2x+1)} = \frac{A}{x-3} + \frac{Cx+D}{6x^2+2x+1}$$

Case IV- Factors of $g(x)$ are repeated quadratic factors.

$$g(x) = (ax+b)(cx^2+dx+e)^n$$

$$\frac{f(x)}{g(x)} = \frac{A}{ax+b} + \frac{C_1x+D_1}{cx^2+dx+e} + \frac{C_2x+D_2}{(cx^2+dx+e)^2} + ... + \frac{C_nx+D_n}{(cx^2+dx+e)^n}$$

Illustration: $$\frac{x^2+5}{(x-2)^2(x^2+2x+3)^3}$$

$$= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C_1x+D_1}{x^2+2x+3} + \frac{C_2x+D_2}{(x^2+2x+3)^2} + \frac{C_3x+D_3}{(x^2+2x+3)^3}$$

Above four possibilities we will discuss here.

The unknown values $A, B, C,$ .... are determined by putting different values of x.

Worked out examples.

Ex 1. $$\frac{2x-3}{x^2+6x+8}$$

$$= \frac{2x-3}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4} = \frac{A(x+4)+B(x+2)}{(x+2)(x+4)}$$

$\Rightarrow 2x-3 = A(x+4) + B(x+2)$

$x = -4 \Rightarrow -11 = -2B \Rightarrow B = 11/2$

$x = -2 \Rightarrow -7 = 2A \Rightarrow A = -7/2$

Hence $$\frac{2x-3}{(x+2)(x+4)} = -\frac{7}{2(x+2)} + \frac{11}{2(x+4)}$$

Ex 2. $$\frac{x^2}{x^2+7x+10} = \frac{x^2+7x+10-(7x+10)}{x^2+7x+10} = 1 - \frac{7x-10}{(x+2)(x+5)}$$

We write $$\frac{7x-10}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5} = \frac{A(x+5)+B(x+2)}{(x+2)(x+5)}$$

$\Rightarrow 7x-10 = A(x+5) + B(x+2)$

$x = -5 \Rightarrow -45 = -3B \Rightarrow B = 15$

$x = -2 \Rightarrow -24 = 3A \Rightarrow A = -8$

Hence $$\frac{7x-10}{(x+2)(x+5)} = -\frac{8}{x+2} + \frac{15}{x+5} \Rightarrow \frac{x^2}{x^2+7x+10} = 1 + \frac{8}{x+2} - \frac{15}{x+4}$$

ii. $2^{\frac{1}{6}}\{\cos(8n+1)\frac{\pi}{12}) + i\sin(8n+1)\frac{\pi}{12}\}$, $n = 0, 1, 2$

iii. $1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

14. Solve: i. $x^3 - 1 = 0$ ii. $x^7 + x^4 + x^3 + 1 = 0$

Ans- i. $1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

ii. $-1, \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}, \pm\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$

15. If $x + \frac{1}{x} = 2\cos\theta$, prove that $x = \cos\theta \pm i\sin\theta$.

Hence show that $x^n + \frac{1}{x^n} = 2\cos n\theta$

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1.2 PARTIAL FRACTION

1.2.1 Introduction: Fraction we consider here is of that type in which numerator and denominator are polynomial function. It is of the form $\frac{f(x)}{g(x)}$. If deg$f(x)$ <deg$g(x)$, then the fraction is **proper fraction**. Otherwise it is an **improper fraction**. An improper fraction can be reduced to sum of a polynomial and a proper fraction. For if deg $g(x)$ <deg $f(x)$, then $f(x) = q(x).g(x) + r(x)$ and

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

So $q(x)$ is a polynomial and $\frac{r(x)}{g(x)}$ is a proper fraction.

If deg $g(x)$ =deg $f(x)$ then $q(x)$ is a constant and is a proper fraction. From elementary algebra, it is known that any number of fractions can be summed up to a single fraction. The inverse process, in which a proper fraction is expressed as the sum of a set of simpler proper fractions is called decomposition of the given fraction into **partial fractions**.

1.2.2 General Rule for resolution into partial fraction :

Let us observe the following sum-

$$\frac{f_1(x)}{g_1(x)} + \frac{f_2(x)}{g_2(x)} + \frac{f_3(x)}{g_3(x)} + \ldots\ldots\ldots\ldots = \frac{f(x)}{g(x)}$$

Here we sum up a number of fractions into a single fraction, and the denominator $g(x)$ of the sum is the LCM of the denominators $g_1(x), g_2(x), g_3(x)$..... Because decomposition is an inverse process hence we require to factorise $g(x)$ into different simple factors. We consider only those $g(x)$ which is factorable to linear and quadratic polynomials.

5. $\dfrac{3x+1}{(x-1)(x^2+1)}$ 6. $\dfrac{x+29}{x^2+8x-9}$ 7. $\dfrac{1}{(x+1)^2(x+2)(x+3)}$ 8. $\dfrac{4x}{(x^2-1)^2}$

9. $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$ 10. $\dfrac{1}{(x^2+x)(x^2-1)}$ 11. $\dfrac{5x+6}{(x-2)(x+1)(2x-3)}$

12. $\dfrac{x}{(x^2+x-2)(x^2-x+2)}$ 13. $\dfrac{x+1}{(x-1)^2(x^2+1)}$ 14. $\dfrac{5+4x}{(1-x)(1+x^2)^2}$

15. $\dfrac{x}{(x-1)(1+x^2)^2}$ 16. $\dfrac{1}{(x^2-1)(x^2+2)}$ 17. $\dfrac{1}{x^3+1}$ 18. $\dfrac{x}{(x-1)(x^3+1)}$

19. $\dfrac{x^3}{(1-x)(1+x)(x^2+4)}$ 20. $\dfrac{2x-5}{(x^2+4)(x-2)(x-3)}$

Answer:

1. $\dfrac{3}{4(x+1)}+\dfrac{1}{4(x-1)}+\dfrac{1}{2(x-1)^2}$ 2. $\dfrac{3}{1-3x}-\dfrac{2}{1-2x}$

3. $-\dfrac{3}{x}+\dfrac{4}{2x-1}+\dfrac{1}{x+1}$ 4. $-\dfrac{1}{9(x+1)}+\dfrac{1}{9(x-2)}+\dfrac{2}{3(x-2)^2}$

5. $\dfrac{2}{x-1}+\dfrac{1-2x}{x^2+1}$ 6. $\dfrac{3}{x-1}-\dfrac{2}{x+9}$

7. $-\dfrac{1}{4(x+1)}+\dfrac{1}{2(x+1)^2}+\dfrac{1}{2(x+2)}-\dfrac{1}{4(x+3)}$ 8. $\dfrac{1}{(x-1)^2}-\dfrac{1}{(x+1)^2}$

9. $-\dfrac{5}{3(x-2)}-\dfrac{4}{(x-2)^2}-\dfrac{1}{3(1-2x)}$ 10. $-\dfrac{1}{x}+\dfrac{1}{4(x-1)}+\dfrac{3}{4(x+1)}+\dfrac{1}{2(x+1)^2}$

11. $\dfrac{7}{x-2}+\dfrac{3}{5(x+1)}-\dfrac{56}{5(2x-3)}$ 12. $\dfrac{1}{8(x+2)}+\dfrac{1}{4(x-1)}-\dfrac{3x-1}{8(x^2-x+2)}$

13. $-\dfrac{1}{2(x-1)}+\dfrac{1}{(x-1)^2}+\dfrac{x-1}{2(x^2+1)}$ 14. $\dfrac{9}{4(1-x)}+\dfrac{9(x+1)}{4(1+x^2)}+\dfrac{9x+1}{2(1+x^2)^2}$

15. $\dfrac{1}{4(x-1)}-\dfrac{x+1}{4(1+x^2)}-\dfrac{x-1}{2(1+x^2)^2}$ 16. $\dfrac{1}{6(x-1)}-\dfrac{1}{6(x+1)}-\dfrac{1}{3(x^2+2)}$

17. $\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$ 18. $\dfrac{1}{2(x-1)}+\dfrac{1}{6(x+1)}-\dfrac{2x-1}{3(x^2-x+1)}$

19. $\dfrac{1}{10(1-x)}-\dfrac{1}{10(1+x)}-\dfrac{4x}{5(x^2+4)}$ 20. $\dfrac{1}{8(x-2)}+\dfrac{1}{13(x-3)}-\dfrac{21x+50}{104(x^2+4)}$

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Hence $\dfrac{x^3}{(x-1)(x-2)(x-3)}=1+\dfrac{6x^2-11x+6}{(x-1)(x-2)(x-3)}$

$$=1+\frac{1}{2(x-1)}-\frac{8}{x-2}+\frac{27}{2(x-3)}$$

Ex 18. $\dfrac{x^4}{(x-1)^4(x+1)}$

$$=\frac{x^4}{(x-1)^4(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}+\frac{E}{x+1}$$

$$=\frac{A(x-1)^3(x+1)+B(x-1)^2(x+1)+C(x-1)(x+1)+D(x+1)+E(x-1)^4}{(x-1)^4(x+1)}$$

$\Rightarrow x^4=A(x-1)^3(x+1)+B(x-1)^2(x+1)+C(x-1)(x+1)$
$+D(x+1)+E(x-1)^4$

$x=1 \Rightarrow 1=2D \qquad \Rightarrow D=1/2$

$x=-1 \Rightarrow 1=16E \qquad \Rightarrow E=1/16$

Equating the coefficients of x^4 we have $1=A+E \qquad \Rightarrow A=15/16$

Equating the coefficients of x^3 we have $0=-2A+B+4E \qquad \Rightarrow B=13/8$

Equating the constant terms $0=-A+B-C+D+E \qquad \Rightarrow C=5/4$

$$\Rightarrow \frac{x^4}{(x-1)^4(x+1)}$$

$$=\frac{15}{16(x-1)}+\frac{13}{8(x-1)^2}+\frac{5}{4(x-1)^3}+\frac{1}{2(x-1)^4}+\frac{1}{16(x+1)}$$

Ex 19. $\dfrac{4x+3}{4x^3+8x^2+3x}$

$$=\frac{4x+3}{x(2x+3)(2x+1)}=\frac{A}{x}+\frac{B}{2x+3}+\frac{C}{2x+1}$$

$$=\frac{A(2x+3)(2x+1)+Bx(2x+1)+Cx(2x+3)}{x(2x+3)(2x+1)}$$

$\Rightarrow 4x+3=A(2x+3)(2x+1)+Bx(2x+1)+Cx(2x+3)$

$x=0 \quad \Rightarrow 3=3A \qquad \Rightarrow A=1$

$x=-3/2 \quad \Rightarrow -3=3B \qquad \Rightarrow B=-1$

$x=-1/2 \quad \Rightarrow 1=-C \qquad \Rightarrow C=-1$

Hence $\dfrac{4x+3}{4x^3+8x^2+3x}=\dfrac{1}{x}-\dfrac{1}{2x+3}-\dfrac{1}{2x+1}$

Ex 20. $\dfrac{4x^3+2x^2+1}{4x^3-x}$

$$=\frac{4x^3-x+2x^2+x+1}{4x^3-x}=1+\frac{2x^2+x+1}{4x^3-x}=1+\frac{2x^2+x+1}{x(2x+1)(2x-1)}$$

We write $\dfrac{2x^2+x+1}{x(2x+1)(2x-1)}=\dfrac{A}{x}+\dfrac{B}{2x+1}+\dfrac{C}{2x-1}$

$$=\frac{A(4x^2-1)+Bx(2x-1)+Cx(2x+1)}{x(2x+1)(2x-1)}$$

$\Rightarrow 2x^2+x+1=A(4x^2-1)+Bx(2x-1)+Cx(2x+1)$

$x=0 \qquad \Rightarrow A=-1$

$x=1/2 \qquad \Rightarrow 2=C$

$x=-1/2 \qquad \Rightarrow 1=B$

Hence $\dfrac{4x^3+2x^2+1}{4x^3-x}=1+\dfrac{2x^2+x+1}{4x^3-x}=1-\dfrac{1}{x}+\dfrac{1}{2x+1}+\dfrac{2}{2x-1}$

Exercise 1.2

Resolve into Partial Fraction-

1. $\dfrac{x^2-x+1}{(x+1)(x-1)^2}$ 2. $\dfrac{1}{(1-2x)(1-3x)}$ 3. $\dfrac{3}{x(2x-1)(x+1)}$ 4. $\dfrac{x}{(x+1)(x-2)^2}$

Equating the coefficients of x we have $0 = A + C = 2 + C \Rightarrow C = -2$
Equating the coefficients of x^2 we have $0 = B$
Equating the constant terms we have $0 = B + D \Rightarrow D = 0$

Hence $\dfrac{2x^3}{(x^2+1)^2} = \dfrac{2x}{x^2+1} - \dfrac{2x}{(x^2+1)^2}$

Ex 15. $\dfrac{x^3+5x^2+x}{(x+1)(x^2+1)(x^3+1)}$

$$= \frac{x^3+5x^2+x}{(x+1)^2(x^2+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{x^2-x+1}$$

$$= \frac{A(x^3+1)(x^2+1) + B(x^2+1)(x^2-x+1) + (Cx+D)(x+1)^2(x^2-x+1)}{(x+1)^2(x^2+1)(x^2-x+1)}$$

$$+ \frac{(Ex+F)(x+1)^2(x^2+1)}{(x+1)^2(x^2+1)(x^2-x+1)}$$

$\Rightarrow x^3 + 5x^2 + x = A(x^3+1)(x^2+1) + B(x^2+1)(x^2-x+1)$
$+ (Cx+D)(x+1)^2(x^2-x+1) + (Ex+F)(x+1)^2(x^2+1)$

Equating the coefficients of x^5 we have $0 = A + C + E$ I
Equating the coefficients of x^4 we have $0 = B + C + D + 2E + F$ II
Equating the coefficients of x^3 we have $1 = A - B + D + 2E + 2F$ III
Equating the coefficients of x^2 we have $5 = A + 2B + C + 2E + 2F$ IV
Equating the coefficients of x we have $1 = -B + C + D + E + 2F$ V
Equating the constant terms we have $0 = A + B + D + F$ VI
From I and IV $2B + E + 2F = 5$ VII
From II and VI $A - C - 2E = 0$ VIII
From III and V $A - C + E = 0$ IX
From VIII and IX $E = 0$
From I $A + C = 0$, From IX $A - C = 0$, $\Rightarrow A = C = 0$
From IV and VI $B + F = 5/2 = -D$ $\Rightarrow D = -5/2$
From V $2F - B = 7/2$, $\Rightarrow 3F = 12/2$ $\Rightarrow F = 2$ $\Rightarrow B = 1/2$

Hence $\dfrac{x^3+5x^2+x}{(x+1)(x^2+1)(x^3+1)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{x^2-x+1}$

$$= \frac{1}{2(x+1)^2} - \frac{5}{2(x^2+1)} + \frac{2}{x^2-x+1}$$

Ex 16. $\dfrac{1}{(x-1)^3(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$$

$$= \frac{A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3}{(x-1)^3(x+1)}$$

$\Rightarrow 1 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3$
$x = 1 \Rightarrow 1 = 2C \Rightarrow C = 1/2$
$x = -1 \Rightarrow 1 = -D \Rightarrow D = -1$
Equating the coefficients of x^3 we have $0 = A + D = A - 1 \Rightarrow A = 1$
Equating the constant terms $1 = A - B + C - D$ $\Rightarrow B = 3/2$

Hence $\dfrac{1}{(x-1)^3(x+1)} = \dfrac{1}{x-1} + \dfrac{3}{2(x-1)^2} + \dfrac{1}{2(x-1)^3} - \dfrac{1}{x+1}$

Ex 17. $\dfrac{x^3}{(x-1)(x-2)(x-3)}$

$$= \frac{x^3-6x^2+11x-6+6x^2-11x+6}{x^3-6x^2+11x-6} = 1 + \frac{6x^2-11x+6}{(x-1)(x-2)(x-3)}$$

We write $\dfrac{6x^2-11x+6}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

$\Rightarrow 6x^2 - 11x + 6 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$
$x = 1 \Rightarrow 1 = 2A \Rightarrow A = 1/2$
$x = 2 \Rightarrow 8 = -B \Rightarrow B = -8$
$x = 3 \Rightarrow 27 = 2C \Rightarrow C = 27/2$

$$=\frac{A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)}{(x+1)(x-1)(x+2)}$$

$\Rightarrow 5x^2-4=A(x-1)(x+2)+B(x+1)(x+2)+C(x+1)(x-1)$

$x=1 \quad \Rightarrow 1=6B \Rightarrow B=1/6$

$x=-1 \quad \Rightarrow 1=-2A \quad \Rightarrow A=-1/2$

$x=-2 \quad \Rightarrow 16=3C \quad \Rightarrow C=16/3$

Hence $\dfrac{x^4}{(x^2-1)(x+2)}=x-2+\dfrac{5x^2-4}{(x+1)(x-1)(x+2)}$

$$=x-2-\frac{1}{2(x+1)}+\frac{1}{6(x-1)}+\frac{16}{3(x+2)}$$

Ex 11. $\dfrac{x}{(x-1)^2(x-2)}$

$$=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}=\frac{A(x-1)(x-2)+B(x-2)+C(x-1)^2}{(x-1)^2(x-2)}$$

$\Rightarrow x=A(x-1)(x-2)+B(x-2)+C(x-1)^2$

$x=1 \quad \Rightarrow 1=-B$

$x=2 \quad \Rightarrow 2=C$

Equating the coefficients of x^2 we have $0=A+C=A+2 \Rightarrow A=-2$

Hence $\dfrac{x}{(x-1)^2(x-2)}=-\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}+\dfrac{2}{x-2}$

Ex 12. $\dfrac{x^3+1}{x^3+4x}$

$$=\frac{x^3+4x-4x+1}{x^3+4x}=1-\frac{4x-1}{x(x^2+4)}$$

We write $\dfrac{4x-1}{x(x^2+4)}=\dfrac{A}{x}+\dfrac{Cx+D}{x^2+4}=\dfrac{A(x^2+4)+(Cx+D)x}{x(x^2+4)}$

$\Rightarrow 4x-1=A(x^2+4)+(Cx+D)x$

$x=0 \quad \Rightarrow -1=4A \quad \Rightarrow A=-1/4$

Equating the coefficients of x^2 we have $0=A+C=-1/4+C \Rightarrow C=1/4$

Equating the coefficients of x we have $4=D$

Hence $\dfrac{x^3+1}{x^3+4x}=1-\dfrac{4x-1}{x(x^2+4)}=1+\dfrac{1}{4x}-\dfrac{x+16}{4(x^2+4)}$

Ex 13. $\dfrac{x^5+9x^3-9x^2-9}{x^3+9x}$

$$=\frac{x^2(x^3+9x)-9x^2-9}{x^3+9x}=x^2-9\frac{x^2+1}{x(x^2+9)}$$

We write $\dfrac{x^2+1}{x(x^2+9)}=\dfrac{A}{x}+\dfrac{Cx+D}{x^2+9}=\dfrac{A(x^2+9)+(Cx+D)x}{x(x^2+9)}$

$\Rightarrow x^2+1=A(x^2+9)+(Cx+D)x$

$x=0 \quad \Rightarrow 1=9A \quad \Rightarrow A=1/9$

Equating the coefficients of x^2 we have $1=A+C=1/9+C \Rightarrow C=8/9$

Equating the coefficients of x we have $0=D$

Hence $\dfrac{x^5+9x^3-9x^2-9}{x^3+9x}=x^2-9\dfrac{x^2+1}{x(x^2+9)}=x^2-9\{\dfrac{1}{9x}+\dfrac{8x}{9(x^2+9)}\}$

$$=x^2-\frac{1}{x}-\frac{8x}{x^2+9}$$

Ex 14. $\dfrac{2x^3}{(x^2+1)^2}$

$$=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}=\frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2}$$

$\Rightarrow 2x^3=(Ax+B)(x^2+1)+(Cx+D)$

Equating the coefficients of x^3 we have $2=A$

Soln: (i) RHS $= {}^{n-1}P_r + r.\,{}^{n-1}P_{r-1} = \dfrac{(n-1)!}{(n-r-1)!} + r \times \dfrac{(n-1)!}{(n-r)!}$

$= \dfrac{(n-1)!}{(n-r-1)!}\left(1+\dfrac{r}{n-r}\right) = \dfrac{(n-1)!}{(n-r-1)!} \times \dfrac{n}{n-r} = \dfrac{n!}{(n-r)!} = {}^{n}P_r =$ LHS

(ii) LHS $= {}^{n+1}P_{r+1} = \dfrac{(n+1)!}{(n-r)!} = (n+1)\dfrac{n!}{(n-r)!} = (n+1)\ {}^{n}P_r =$ RHS

(iii) LHS $= {}^{n}P_r = \dfrac{n!}{(n-r)!} = n(n-1)\ \dfrac{(n-2)!}{\{n-2-(r-2)\}!} = n(n-1)\ {}^{n-2}P_{r-2}$

= RHS

(iv) LHS $= 1.\,{}^{1}P_1 + 2.\,{}^{2}P_2 + 3.\,{}^{3}P_3 + \ldots\ldots\ldots\ldots + n.\,{}^{n}P_n$

$= (2-1)\,{}^{1}P_1 + (3-1)\,{}^{2}P_2 + (4-1)\,{}^{3}P_3 + \ldots\ldots\ldots + (n+1-1)\,{}^{n}P_n$

$= 2.\,{}^{1}P_1 - {}^{1}P_1 + 3.\,{}^{2}P_2 - {}^{2}P_2 + 4.\,{}^{3}P_3 - {}^{3}P_3 + \ldots\ldots\ldots + (n+1)\,{}^{n}P_n - {}^{n}P_n$

(It is observed that $(n+1)\,{}^{n}P_n = (n+1)n! = (n+1)! = {}^{n+1}P_{n+1}$)

$= {}^{2}P_2 - {}^{1}P_1 + {}^{3}P_3 - {}^{2}P_2 + {}^{4}P_4 - {}^{3}P_3 + \ldots\ldots\ldots + {}^{n+1}P_{n+1} - {}^{n}P_n$

$= {}^{n+1}P_{n+1} - 1 =$ RHS

(v) LHS $= (2n)! = 2n(2n-1)(2n-2)(2n-3)\ldots\ldots\ldots 3.2.1$

$= \{2n(2n-2)(2n-4)\ldots\ldots\ldots 4.2\}\{(2n-1)(2n-3)\ldots\ldots\ldots 3.1\}$

$= 2^n\{n(n-1)(n-2)(n-3)\ldots\ldots\ldots 3.2.1\}\{(2n-1)(2n-3)\ldots\ldots\ldots 3.1\}$

$= 2^n n!\,\{1.3.5\ldots\ldots\ldots(2n-1)\} =$ RHS

Now ${}^{2n}P_n = \dfrac{(2n)!}{(2n-n)!} = \dfrac{(2n)!}{n!} = \dfrac{2^n n!\{1.3.5\ldots\ldots\ldots(2n-1)\}}{n!}$

$= 2^n\{1.3.5\ldots\ldots\ldots(2n-1)\}$ Proved.

Ex 4. [QP 2013] In how many ways the letters of the word DEER can be arranged?

Soln: In the word DEER there are altogether 4 letters. Out of these 4 letters E occurs 2 times. So the total number of possible arrangements is $\dfrac{4!}{2!} = 12$

Ex 5. In how many ways can the letters of the word MATHEMATICS be arranged?

Soln: In the word MATHEMATICS there are altogether 11 letters. Out of these 11 letters M occurs 2 times, A occurs 2 times, T occurs 2 times. So the total number of possible arrangements is $\dfrac{11!}{2!2!2!} = 4989600$

Ex 6. In how many ways the letters of the word DAUGHTER be arranged so that the vowels may never be separated?

Soln: In the word DAUGHTER we will take 3 vowels A, U, E as one letter. Other letters are D, G, H, T, R. Therefore altogether total letters are D, G, H, T, R, (A,U,E). i.e. 6 letters to arrange. Number of arrangements is ${}^{6}P_6 = 6! = 720$.

Again the vowels can be arranged (among themselves) in ${}^{3}P_3 = 6$ ways.

Therefore total number of arrangements is $720 \times 6 = 4320$

Ex 7. How many words can be formed out of the letters of the word GUWAHATI by keeping the vowels together?

Soln: There are 4 vowels A, A, U, I in the word GUWAHATI. Let the 4 vowels be taken as 1 letter. Then the total number of letters is 5 i.e. G, W, H, T, (A, A, U, I)

These letters can be arranged among themselves in ${}^{5}P_5 = 120$ ways. Again the vowels can be arranged (among themselves) in $\dfrac{4!}{2!} = 12$ ways. Therefore total number of arrangements is $120 \times 12 = 1440$.

Ex 8. In how many ways can the letters of the word PREUNIVERSITY be arranged?

Soln: In PREUNIVERSITY total number of letters is 13. Here E occurs 2 times, R occurs 2 times, I occurs 2 times.

Therefore required number of arrangements is $\dfrac{13!}{2!2!2!}$

(v) $^{10}P_4 = \frac{10!}{6!} = 10\times9\times8\times7 = 5040$ (vi) $^8P_5 = \frac{8!}{3!} = 8\times7\times6\times5\times4 = 6720$

Ex 2. Find n - (i) **[QP 2010]** $^nP_4 = 10\times{}^nP_3$ (ii) $^{33}P_n = 19\times{}^{33}P_{n-1}$

(iii) $^nP_4 : {}^{n-1}P_3 = 9:1$ (iv) $^{n+1}P_6 : {}^{n-1}P_7 = 5:12$

(v) $^nP_3 : {}^{n+2}P_3 = 5:12$ (vi) $^nP_6 = 30\times{}^nP_4$

Soln: (i) $^nP_4 = 10\times{}^nP_3$

$$\Rightarrow \frac{n!}{(n-4)!} = 10\times\frac{n!}{(n-3)!} \quad \Rightarrow \frac{n!}{(n-4)!} = 10\times\frac{n!}{(n-3)\cancel{(n-4)!}}$$

$$\Rightarrow 1 = \frac{10}{n-3} \quad \Rightarrow n = 13$$

(ii) $^{33}P_n = 19\times{}^{33}P_{n-1}$

$$\Rightarrow \frac{33!}{(33-n)!} = 19\times\frac{33!}{(33-n+1)!} \quad \Rightarrow \frac{33!}{(33-n)!} = 19\times\frac{33!}{(34-n)(33-n)!}$$

$$\Rightarrow 1 = \frac{19}{34-n} \quad \Rightarrow n = 15$$

(iii) $^nP_4 : {}^{n-1}P_3 = 9:1$

$$\Rightarrow \frac{n!}{(n-4)!} : \frac{(n-1)!}{(n-4)!} = 9:1 \Rightarrow \frac{n!}{(n-1)!} = 9 \quad \Rightarrow n = 9$$

(iv) $^{n+1}P_6 : {}^{n-1}P_7 = 5:12$

$$\Rightarrow \frac{(n+1)!}{(n+1-6)!} : \frac{(n-1)!}{(n-1-7)!} = 5:12 \quad \Rightarrow \frac{(n+1)!}{(n-5)!} : \frac{(n-1)!}{(n-8)!} = 5:12$$

$$\Rightarrow \frac{(n+1)!(n-8)!}{(n-5)!(n-1)!} = 5:12 \quad \Rightarrow \frac{(n+1)n}{(n-5)(n-6)(n-7)} = \frac{5}{12}$$

$$\Rightarrow 12n(n+1) = 5(n-5)(n-6)(n-7)$$

$$\Rightarrow 5n^3 - 102n^2 + 523n - 1050 = 0$$

$$\Rightarrow (n-14)(5n^2-32n+75) = 0$$

$5n^2 - 32n + 75 = 0$ does not have a real root. Only integral root is $n = 14$.

(v) $^nP_3 : {}^{n+2}P_3 = 5:12$

$$\Rightarrow \frac{n!}{(n-3)!} : \frac{(n+2)!}{(n+2-3)!} = 5:12 \quad \Rightarrow \frac{n!}{(n-3)!} : \frac{(n+2)!}{(n-1)!} = 5:12$$

$$\Rightarrow \frac{n!(n-1)!}{(n-3)!(n+2)!} = 5:12 \quad \Rightarrow \frac{(n-1)(n-2)}{(n+2)(n+1)} = \frac{5}{12}$$

$$\Rightarrow 12(n-1)(n-2) = 5(n+1)(n+2)$$

$$\Rightarrow 7n^2 - 51n + 14 = 0$$

$\Rightarrow (n-7)(7n-2) = 0$ We need an integral root. Hence $n = 7$.

(vi) $^nP_6 = 30\times{}^nP_4$

$$\Rightarrow \frac{n!}{(n-6)!} = 30\times\frac{n!}{(n-4)!} \quad \Rightarrow \frac{n!}{(n-6)!} = 30\times\frac{n!}{(n-4)(n-5)(n-6)!}$$

$$\Rightarrow 1 = \frac{30}{(n-4)(n-5)} \quad \Rightarrow (n-4)(n-5) = 30$$

$$\Rightarrow n^2 - 9n + 20 = 30$$

$$\Rightarrow n^2 - 9n - 10 = 0 \quad \Rightarrow (n-10)(n+1) = 0$$

Since n should be a positive integer, hence $n = 10$.

Ex 3. Prove that (i) $^nP_r = {}^{n-1}P_r + r\,.\,{}^{n-1}P_{r-1}$

(ii) $^{n+1}P_{r+1} = (n+1)\,{}^nP_r$

(iii) $^nP_r = n(n-1)\,{}^{n-2}P_{r-2}$

(iv) $1.{}^1P_1 + 2.{}^2P_2 + 3.{}^3P_3 + \ldots\ldots + n.{}^nP_n = {}^{n+1}P_{n+1} - 1$

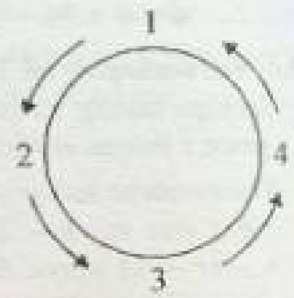
(v) **[QP 2012]** $(2n)! = 2^n n!\{1.3.5\ldots\ldots(2n-1)\}$

Hence show that $^{2n}P_n = 2^n\{1.3.5\ldots\ldots(2n-1)\}$

permutation is $\frac{4!}{2!}$. In the word DISCUSS **3** letters are alike. Here total number of different words formed by the letters is $\frac{7!}{3!}$. Suppose there are n things. Let these n things be represented by n letters and suppose p of them to be a, q of them to be b, r of them to be c and the rest are all different.

Number of permutation of these n letters taking all at a time is $\frac{n!}{p!q!r!}$.

1.3.6 Permutation in a Ring or Circle: In Circular permutation there is no end or beginning. For example consider four numbers 1, 2, 3, 4. Starting from each number we have the arrangements 1234, 2341, 3412, 4123. But all these are same in circular permutation. So 4 numbers are arranged in a straight line in 4! ways. And these 4 numbers we can arrange in a circle in $\frac{4!}{4}=3!$ ways. Thus the number of ways in which n different things can be arranged in a circle is $(n-1)!$. If the distinction between the clockwise and counter-clockwise arrangement be not made as in case of weaving flowers to a garland the number of permutation is $\frac{1}{2}(n-1)!$



1.3.7 Combination : A selection that can be formed by taking some or all of a finite set of things (or objects) is called **Combination**. The number of combination of n things taken r at a time is denoted by ${}^nC_r$

Theorem: The number of combination of n different things taken r at a time is

$${}^nC_r=\frac{n!}{r!(n-r)!}$$

Proof: Let ${}^nC_r$ be the required number of combinations. Since every combination of r different things produces $r!$ permutations when the r things are arranged in all possible ways, ${}^nC_r$ combinations, each combination containing r different things, will produce ${}^nC_r\times r!$ permutations. Again number of permutations of n different things taken r at a time is ${}^nP_r$

Hence $\quad {}^nC_r\times r!={}^nP_r=\frac{n!}{(n-r)!}$

Thus $\quad {}^nC_r=\frac{n!}{r!(n-r)!}$

e.g. (i) Choosing 3 desserts from a menu of 10 is ${}^{10}C_3=120$

(ii) Picking a team of 4 people from a group of $12={}^{12}C_4=495$

1.3.8 Deduction : (i) ${}^nC_1=\frac{n!}{1!(n-1)!}=\frac{n(n-1)!}{(n-1)!}=n$

(ii) ${}^nC_n=\frac{n!}{n!(n-n)!}=\frac{n!}{n!}=1$

(iii) ${}^nC_r\times r!={}^nP_r$

Worked out Examples :

Ex 1. Find value of (i) ${}^6P_3$ (ii) ${}^8P_0$ (iii) ${}^{11}P_1$ (iv) **[QP 2011]** $\frac{8!}{6!2!}$

(v) **[QP 2015]** ${}^{10}P_4$ (vi) **[QP 2016]** ${}^8P_5$

Soln: (i) ${}^6P_3=\frac{6!}{(6-3)!}=6.5.4=120$

(ii) ${}^8P_0=\frac{8!}{(8-0)!}=1$

(iii) ${}^{11}P_1=\frac{11!}{(11-1)!}=11$

(iv) $\frac{8!}{6!2!}=\frac{8\times7}{2}=28$

1.3 PERMUTATION AND COMBINATION

1.3.1 An arrangement that can be formed by taking some or all of a finite set of things (or objects) is called a **Permutation.** Order of the things is very important in case of permutation. A permutation is said to be a **Linear Permutation** if the objects are arranged in a line. A linear permutation is simply called as a permutation. In other word, Permutation can be defined as the different arrangements which can be made out of a given set of things by taking some or all of them at a time. Permutation of three letters P, Q and R taking one, two or three at a time are respectively.

P	PQ	PQR
Q	QR	QRP
R	RP	RPQ
	PR	PRQ
	RQ	QPR
	QP	RQP

1.3.2 Factorial Notation : The continuous product of the first 'n' natural numbers is called factorial n and is denoted by n!.

i.e, $n! = 1\times2\times3\times \ldots\ldots \times(n-1)\times n.$

$= n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\ldots 3.2.1$

$= n\times(n-1)!$

Thus $5! = 5.4.3.2.1 = 120,\ 3! = 3.2.1 = 6$. etc.

We have, $n\times(n-1)! = n!$

Hence $n = \dfrac{n!}{(n-1)!}$ Put $n = 1$, we have, $1 = \dfrac{1!}{0!}$ So we define $0! = 1$

1.3.3 Fundamental Principle: If one operation can be performed independently in m different ways and if corresponding to each way a second operation can be performed in n different ways then the two operations can be performed together in $m \times n$ ways. If P, Q, R are placed in two locations, in first place we keep in 3

different ways- P, Q or R. In the second place Q or R comes with P, P or Q comes with R, and lastly P or R comes with Q. Each of 3 ways for the first place are associated with 2 different ways for the second location. Thus permutation of three letters taken two at a time is 3×2.

Example: A cinema hall has 6 doors. In how many ways can a man enter and (i) leave it by different door? (ii) leave it by any door?

(i) The man can enter the hall in 6 different ways, and corresponding to each of these 6 ways he can leave the hall in 5 ways since he cannot go out through the door he entered. Required number of permutation is $6 \times 5 = 30$.

(ii) He can go out through any of 6 doors. Required number of permutation is $6 \times 6 = 36$.

1.3.4 Permutation of Things all different: n distinct things are there. We are to arrange taking r things condition is that things will not repeat.. Means we are to place r things at r locations:

The extreme left position is filled in n ways. Next one is filled in $n-1$ ways. Next one in $n-2$ ways and goes on. The r^{th} location is filled in $n-(r-1)$ ways. So the required number of permutation is $n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\{n-(r-1)\}$

It is denoted by ${}^nP_r$. Now we have

$${}^nP_r = n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\{n-(r-1)\} = \frac{n!}{(n-r)!}$$

Or $${}^nP_r = \frac{n!}{(n-r)!}$$

Permutation of n things taken all at a time is

$${}^nP_n = n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots 3.2.1 = n! = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

This establishes the necessity of defining 0! as 1.

1.3.5 Permutation of Things when they are not all different : BOOK is a 4 letter word. We are asked to form different words with these letters. If O and O are O_1 and O_2, then they are different and two arrangements BKO_1O_2 and BKO_2O_1 are different. Instead the two O's replace for same word. Here required number of

Hence $^{n}C_{14} = {}^{15}C_{14} = 15$

Ex 27. Prove that $^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = 0$

Soln: $^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

$= ({}^{15}C_8 + {}^{15}C_9) - ({}^{15}C_6 + {}^{15}C_7)$

$= {}^{16}C_9 - {}^{16}C_7$ since $^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

$= 0$ since $^{n}C_r = {}^{n}C_{n-r}$

Ex 28. In how many ways can 7 books be selected from 10 books?

Soln: Number of arrangements that 7 books be selected from 10 books is equal to the number of combinations of 10 things taken 7 at a time.

So 7 books be selected from 10 books in $^{10}C_7 = \frac{10!}{3!7!} = 120$ ways.

Ex 29. How many groups of 12 persons can be formed out of 16 persons?

Soln: Out of 16 persons number of groups can be formed consisting of 12 persons is

$$^{16}C_{12} = \frac{16!}{4!12!} = 1820$$

Ex 30. [QP 2014] How many chords can be drawn through 11 points on a circle?

Soln: No three points from the points on a circle are collinear. So out of 11 points every 2 points make a distinct chord. Number chords that can be formed from 11 points on a circle is $^{11}C_2 = 55$.

Ex 31. [QP 2013, 2016] Out of 9 girls and 13 boys how many different committees can be formed, each consisting of 5 boys and 3 girls?

Soln: 5 boys can be selected out of 13 boys in $^{13}C_5 = \frac{13!}{5!8!} = 1287$ ways.

3 girls can be selected out of 9 girls in $^{9}C_3 = \frac{9!}{3!6!} = 84$ ways.

Required number of combination is $^{13}C_5 \times {}^{9}C_3 = 1287 \times 84 = 108108$

Ex 32. In how many ways 12 different books can be distributed equally among 3 students?

Soln: Each student will get 4 books.

The first student may get 4 books in $^{12}C_4 = 495$ ways.

After it is done the second student may get 4 books in $^{8}C_4 = 70$ ways.

The remaining 4 books will be given to the third student in $^{4}C_4 = 1$ way.

Required number of combination is

$$^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4 = 495 \times 70 \times 1 = 34650$$

Ex 33. How many different ways can 9 different fruits be divided into 3 packets of 1, 3 and 5 fruits?

Soln: In first packet 1 fruit is selected in $^{9}C_1 = 9$ ways.

In the second packet 3 fruits can be selected out of 8 fruits in $^{8}C_3 = 56$ ways.

In the third packet 5 fruits can be selected out of 5 remaining fruits in $^{5}C_5 = 1$ way.

Required number of combination is $^{9}C_1 \times {}^{8}C_3 \times {}^{5}C_5 = 9 \times 56 \times 1 = 504$

Ex 34. A cricket team consisting of 11 players is to be selected from 2 groups consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the group of 6 shall contribute no fewer than 4 players?

Soln: Group A consists of 6 players. Group B consists of 8 players.

Case 1… 4 from group A and 7 from group B.

Number of combination is $^{6}C_4 \times {}^{8}C_7 = 15 \times 8 = 120$

Case 2… 5 from group A and 6 from group B.

Number of combination is $^{6}C_5 \times {}^{8}C_6 = 6 \times 28 = 168$

Case 3… 6 from group A and 5 from group B.

Number of combination is $^{6}C_6 \times {}^{8}C_5 = 1 \times 56 = 56$

Required number of combination is $120 + 168 + 56 = 344$

$$= \frac{\{2(2n)2(2n-1)2(2n-2).....\}\{1.3.5........(4n-1)\}}{(2n)!(2n)!}$$

$$= \frac{2^{2n}(2n)!\{1.3.5........(4n-1)\}}{(2n)!(2n)!} = \frac{2^{2n}\{1.3.5.........(4n-1)\}}{(2n)!}$$

$$^{2n}C_n = \frac{(2n)!}{n!n!} = \frac{(2n)(2n-1)(2n-2)......4.3.2.1}{n!n!}$$

$$= \frac{\{2(n)2(n-1)2(n-2).....2.1\}\{1.3.5......(2n-1)\}}{n!n!}$$

$$= \frac{2^n n!\{1.3.5.........(2n-1)\}}{n!n!} = \frac{2^n\{1.3.5........(2n-1)\}}{n!}$$

$$= \frac{2^{2n}\{1.3.5...........(2n-1)\}}{2^n n!} = \frac{2^{2n}\{1.3.5........(2n-1)\}}{(2n)(2n-2)(2n-4)....6.4.2}$$

$$= \frac{2^{2n}\{1.3.5........(2n-1)\}\{1.3.5........(2n-1)\}}{\{(2n)(2n-2)(2n-4)....6.4.2\}\{1.3.5........(2n-1)\}}$$

$$= \frac{2^{2n}\{1.3.5.......(2n-1)\}^2}{(2n)!}$$

Now we have $^{4n}C_{2n} : {}^{2n}C_n$

$$= \left(\frac{2^{2n}\{1.3.5........(4n-1)\}}{(2n)!}\right) / \left(\frac{2^{2n}\{1.3.5.......(2n-1)\}^2}{(2n)!}\right)$$

$$= \frac{1.3.5........(4n-1)}{\{1.3.5........(2n-1)\}^2}$$

$$= \{1.3.5..........(4n-1)\} : \{1.3.5..........(2n-1)\}^2$$

Ex 25. Find n (i) **[QP 2010]** $^nC_{20} = {}^nC_4$ (ii) **[QP 2016]** $^{17}C_5 = {}^nC_{12}$

(iii) **[QP 2011]** $^nC_{12} = {}^nC_8$ (iv) **[QP 2015]** $^nC_2 = {}^nC_{20}$

(v) **[QP 2011]** $^nC_5 = {}^nC_{12}$

Soln: (i) $^nC_{20} = {}^nC_{n-20} = {}^nC_4 \Rightarrow n-20=4 \Rightarrow n=24$

(ii) $^{17}C_5 = {}^{17}C_{17-5} = {}^{17}C_{12} \Rightarrow n=17.$

(iii) $^nC_{12} = {}^nC_{n-12} = {}^nC_8 \Rightarrow n-12=8 \Rightarrow n=20$

(iv) $^nC_2 = {}^nC_{n-2} = {}^nC_{20} \Rightarrow n-2=20 \Rightarrow n=22$

(v) $^nC_5 = {}^nC_{n-5} = {}^nC_{12} \Rightarrow n-5=12 \Rightarrow n=17$

Ex 26. (i) If $^nP_r = 110$, $^nC_r = 55$, find r.

(ii) **[QP 2011]** If $^nP_3 = 336$, find $^nC_3$

(iii) **[QP 2009]** If $^{2n}C_r = {}^{2n}C_{r+2}$ find r

(iv) **[QP 2012, 2014]** If $^{2n}C_3 : {}^nC_2 = 12:1$ find n

(v) **[QP 2010]** If $^nC_{10} = {}^nC_5$ find $^nC_{14}$

Soln: (i) $^nC_r = \frac{1}{r!} \times {}^nP_r \Rightarrow 55 = \frac{1}{r!} \times 110 \Rightarrow r=2$

(ii) $^nC_3 = \frac{1}{3!} \times {}^nP_3 = \frac{336}{6} = 56$

(iii) $^{2n}C_r = {}^{2n}C_{r+2} = {}^{2n}C_{2n-r}$

$\Rightarrow 2n - r = r+2 \Rightarrow r = n-1$

(iv) $^{2n}C_3 : {}^nC_2 = \left(\frac{(2n)!}{3!(2n-3)!}\right) / \left(\frac{n!}{2!(n-2)!}\right)$

$$= \frac{(2n)!2!(n-2)!}{n!3!(2n-3)!} = \frac{(2n)(2n-1)(2n-2)}{3n(n-1)} = \frac{4(2n-1)}{3}$$

So we have $\frac{4(2n-1)}{3} = 12 \Rightarrow 2n=10 \Rightarrow n=5$

v) $^nC_{10} = {}^nC_5 = {}^nC_{n-10} \Rightarrow n-10=5 \Rightarrow n=15$

Therefore required number of arrangement is $5! = 120$.

Ex 20. In how many ways 7 men and 3 women can sit in a round table such that women are always together?

Soln: Because 3 women are together so they are considered as 1 individual. 7 men along with this 1 can be arranged in a round table in $7!$ ways. 3 women can be arranged in $3!$ ways. Required number of arrangement is $7! \times 3! = 30240$.

Ex 21. [QP 2009] In how many ways 8 question papers be arranged so that the best and the worst papers never come together?

Soln: 8 examination papers can be arranged in $8!$ ways. Taking the best and the worst scripts together i.e. considering them as one single script the 7 examination papers be arranged in $7!$ ways. Again the best and worst paper can arrange themselves in $2!$ ways. Thus the number of arrangements is $7! \times 2!$ such that the best and the worst remain together. Thus out of $8!$ arrangements, $7! \times 2!$ arrangements are for keeping best and the worst remain together.

Thus required arrangements that the best and the worst are never together is $8! - 7! \times 2! = 7! \times 6 = 30240$.

Ex 22. [QP 2015] In how many ways 5 boys and 3 girls can be arranged in a row such that no two girls are kept together?

Soln: 5 boys are arranged as $5! = 120$ different ways. 3 girls are placed in 6 different places such that boys come between 2 girls. Number of arrangements is ${}^6P_3 = 120$. Total number of permutation is $120 \times 120 = 14400$.

Ex 23. [QP 2016] A railway has 13 stations on its line. How many different single tickets of each class is necessary?

Solution: One ticket marks 2 stations- one up, the other down.

Number of tickets is ${}^{13}P_2 = 156$.

Ex 24. Prove that (i) ${}^nC_r = {}^nC_{n-r}$ (ii) **[QP 2014]** ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_r + 2.\,{}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$

(iv) ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots\dots\dots + {}^rC_r = {}^{n+1}C_{r+1}$

(v) ${}^{4n}C_{2n} : {}^{2n}C_n = \{1.3.5.\dots\dots\dots(4n-1)\} : \{1.3.5.\dots\dots\dots(2n-1)\}^2$

Proof: (i) $${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n!}{\{n-(n-r)\}!(n-r)!} = {}^nC_{n-r}$$

It is also called complementary combination.

(ii) $${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} = \frac{n!(n-r+1+r)}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!(n+1-r)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

(iii) $${}^nC_r + 2.\,{}^nC_{r-1} + {}^nC_{r-2} = ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$$

$= {}^{n+1}C_r + {}^{n+1}C_{r-1}$ from result (ii) above

$= {}^{n+2}C_r$

(iv) From (ii) ${}^nC_r = {}^{n+1}C_{r+1} - {}^nC_{r+1}$

${}^{n-1}C_r = {}^nC_{r+1} - {}^{n-1}C_{r+1}$

${}^{n-2}C_r = {}^{n-1}C_{r+1} - {}^{n-2}C_{r+1}$

....

....

${}^{r+1}C_r = {}^{r+2}C_{r+1} - {}^{r+1}C_{r+1} = {}^{r+2}C_{r+1} - 1$

Now we have

${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots\dots\dots + {}^rC_r$

$= ({}^{n+1}C_{r+1} - {}^nC_{r+1}) + ({}^nC_{r+1} - {}^{n-1}C_{r+1}) + ({}^{n-1}C_{r+1} - {}^{n-2}C_{r+1}) + \dots\dots$
$+ \dots\dots\dots\dots + ({}^{r+2}C_{r+1} - 1) + {}^rC_r$

$= {}^{n+1}C_{r+1}$ since ${}^rC_r = 1$

(v) $${}^{4n}C_{2n} = \frac{(4n)!}{(2n)!(2n)!} = \frac{(4n)(4n-1)(4n-2)\dots\dots4.3.2.1}{(2n)!(2n)!}$$

Ex 9. [QP 2015] In how many ways can the letters of the word POLYTECHNIC be arranged keeping the vowels together?

Soln: There are 3 vowels O, E, I in the word POLYTECHNIC. Let the 3 vowels be taken as 1 letter. Then the total number of letters is 9 i.e. P, L, Y, T, C, H, N, C, (O, E, I) where C's are 2. These letters can be arranged among themselves in

$\frac{9!}{2!} = 181440$ ways. Again the vowels can be arranged (among themselves) in $3! = 6$ ways. Therefore total number of arrangements is

$181440 \times 6 = 1088640$.

Ex 10. [QP 2014] In how many ways can the letters of the word MULTIPLE be arranged without changing the order of the vowels in the word?

Soln: There are 3 vowels- U, I, E. Arrangement of U, I, E is only UIE since order should not be changed. So for the 3! arrangements of U, I, E we consider only 1 arrangement. Also for the 2 L we consider 2! arrangements as 1. Hence

required number of arrangements is $\frac{8!}{3!2!} = 3360$.

Ex 11. In how many ways can the letters of the word EDUCATION be arranged?.

Soln: There are 9 different letters. So number of permutation is ${}^9P_9 = 9!$.

Ex 12. [QP 2014] 6 different colours are chosen to make a tri-colour flag. How many different flags can be made?

Soln: Number of different flags is ${}^6P_3 = 120$.

Ex 13. [QP 2013] How many odd numbers of 5 distinct significant digits can be formed with 0, 1, 2, 3, 4?

Soln: Digits will not repeat since all digits will be distinct.

Unit place is filled in 2 ways- by 1 or 3. 0 will not go to ten thousand place. So it is filled in 3 ways, after placement of unit place. The middle places are filled in 3! ways. So number of permutation is $2\times3\times3! = 36$.

Ex 14. How many 4 digit numbers divisible by 5 can be formed using 0,2, 5 and 7?

Soln: Since the number is divisible by 5, so unit place is filled by 0 or 5.

If it is 0 then the other places are filled in $3! = 6$ ways.

If unit place digit is 5, then thousand place is filled in 2 ways, and ten and hundred places are placed in 2! ways. In this case number of permutation is $2\times2! = 4$.

Total number of arrangements is $6 + 4 = 10$.

Ex 15. How many integers between 1000 and 10000 have no digits other than 4, 5, 6?

Soln: It means the numbers are 4 digit numbers with 4, 5 and 6. Each of the 4 places is filled in 3 ways. So number of permutation is $3\times3\times3\times3 = 81$.

Ex 16. How many different numbers of 4 distinct digits greater than 5000 can be formed with 3, 4, 5, 6, 7?

Soln: The thousand place is filled in 3 ways- by 5, 6 or 7. After that hundred place is filled in 4 ways, ten place in 3 ways and unit place in 2 ways.

Number of permutation is $3\times4\times3\times2 = 72$.

Ex 17. A child has 3 pockets and 4 coins. In how many ways can he put the coins in his pockets?

Soln: First coin can be put in 3 ways, similarly the second, third and fourth coins also can be put in 3 ways.

So total number of ways $= 3\times3\times3\times3 = 81$.

Ex 18. In how many ways 5 mathematics books and 3 science books can be arranged in a shelf so that no two science books are always together.

Soln: 5 mathematics books can be arranged in 5! ways.

Between the mathematics books there are 4 gaps. If the 3 science books are placed in these 4 gaps or at the 2 ends then no two science book will come together. So we find 6 places to keep 3 science books. For this number of arrangements is ${}^6P_3 = 120$.

Therefore the number of arrangement that no two science book are always together is $5! \times 120 = 14400$

Ex 19. In how many ways can 6 boys form a ring?

Soln: Let the boys be A, B, C, D, E, F.

Let A stand up in certain position. Then the remaining 5 can be arranged in 5! ways among themselves.

In the expansion $(3-x)^7$ the total number of term is $7+1=8$

6^{th} term is $T_6=(-1)^5\,{}^7C_5\,3^2x^5=-{}^7C_5\,3^2x^5$

1.4.2 Middle term : The number of terms in the expansion $(a+x)^n$ is $n+1$.

Case I- If n is even then $n+1$ is odd. In this case there is only one middle term.

Now, let $n=2m$ i.e $m=\frac{n}{2}$ Since number of terms in the expansion $(a+x)^n$ is $n+1=2m+1$ here only middle term is $(m+1)^{th}$ term.

Middle term is $T_{m+1}={}^nC_m\,a^{n-m}x^m$

Case II- If n is odd then $n+1$ is even. In this case there are two middle terms.

Now, let $n=2m+1$ i.e $m=\frac{n-1}{2}$ Since number of terms in the expansion $(a+x)^n$ is $n+1=2m+2$ here middle terms are $(m+1)^{th}$ term and $(m+2)^{th}$ term. Middle terms are $T_{m+1}={}^nC_m\,a^{n-m}x^m$ and $T_{m+2}={}^nC_{m+1}\,a^{n-m-1}x^{m+1}$

1.4.3 Equidistant Terms: In the expansion $(a+x)^n$ the co-efficient of the term equidistant from the beginning and the end are equal. The co-efficient of x^r i.e $(r+1)^{th}$ term from the beginning is ${}^nC_r$. Again the co-efficient of x^{n-r} i.e $(r+1)^{th}$ term from the end is ${}^nC_{n-r}$. But, ${}^nC_r={}^nC_{n-r}$ Hence.

1.4.4 Greatest co-efficient : In the expansion $(a+x)^n$, when n is even the middle term has the greatest co-efficient. When n is odd the two middle terms has the greatest co-efficient which are equal.

1.4.5 Binomial Theorem for a negative integer and fraction :

When n is not a positive integer then the expansion is infinite. In this case the expansion is defined for $1+x$ where $|x|<1$. Thus

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-3)}{3!}x^3+\dots\dots\dots\dots$$
$$+\frac{n(n-1)(n-2)\dots\dots\dots(n-r+1)}{r!}x^r$$
$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Note :1. $(1-x)^n=1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-3)}{3!}x^3+\dots\dots\dots\dots$

$$+(-1)^r\frac{n(n-1)(n-2)\dots\dots\dots(n-r+1)}{r!}x^r+\dots\dots\dots\dots$$

2. $(1+x)^{-1}=1-x+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3+\dots\dots\dots\dots$

$$=1-x+x^2-x^3+x^4-\dots\dots\dots\dots$$

3. $(1-x)^{-1}=1+x+x^2+x^3+x^4+\dots\dots\dots\dots$

4. $(a+x)^n=a^n(1+\frac{x}{a})^n$

$$=a^n\{1+n(\frac{x}{a})+\frac{n(n-1)}{2!}(\frac{x}{a})^2+\frac{n(n-1)(n-3)}{3!}(\frac{x}{a})^3$$
$$+\frac{n(n-1)(n-3)(n-4)}{4!}(\frac{x}{a})^4+\dots\dots\dots\dots\}$$

Worked out examples :

Ex 1. Using binomial theorem expand (i) $(2x+\frac{1}{2x})^5$ (ii) $(4a+3b^2c)^6$

Soln: (i) $(2x+\frac{1}{2x})^5={}^5C_0(2x)^5(\frac{1}{2x})^0+{}^5C_1(2x)^{5-1}(\frac{1}{2x})^1+{}^5C_2(2x)^{5-2}(\frac{1}{2x})^2$

$$+{}^5C_3(2x)^{5-3}(\frac{1}{2x})^3+{}^5C_4(2x)^{5-4}(\frac{1}{2x})^4+{}^5C_5(2x)^{5-5}(\frac{1}{2x})^5$$

$$=2^5x^5+5\times2^3x^3+10\times2x+10\times\frac{1}{2x}+5\times\frac{1}{2^3x^3}+\frac{1}{2^5x^5}$$

$$=32x^5+40x^3+20x+\frac{5}{x}+\frac{5}{8x^3}+\frac{1}{32x^5}$$

(ii) $(4a-3b^2c)^6={}^6C_0(4a)^6+{}^6C_1(4a)^5(-3b^2c)+{}^6C_2(4a)^4(-3b^2c)^2$

37. In a Mathematics paper there are 10 questions and a student is to answer 6 questions. In how many ways can he answer? Ans: 210

38. How many different committees each consisting of 6 members can be formed out of 8 gentlemen and 3 ladies so as to include at least one lady in each committee? Ans: 434

39. In an examination paper 11 questions are set. In how many different ways can one choose 6 questions to answer. If however no 11 is made compulsory in how many ways can he select to answer 6 questions in all? Ans: ${}^{11}C_6$, ${}^{10}C_5$

40. There are 10 points in a plane of which 4 points are collinear. By connecting these points how many triangles can be obtained? Ans: 116

41. There are 20 male teachers and 8 female teachers in a school. A committee is to be formed by taking 5 male teachers and 3 female teachers. In how many ways this can be done? Ans: ${}^{20}C_5 \times {}^{8}C_3$

42. In how many ways can a team of 11 players be selected from 14 football players where 2 of them can play as goalkeeper only? Ans: 132

43. A team of 8 players is to be selected out of 8 boys and 6 girls. If exclusion of a particular girl and inclusion of a particular boy is certain in how many ways can the team be selected if it contains 3 girls? Ans: ${}^{7}C_4 \times {}^{5}C_3$

* * * * *

1.4 BINOMIAL THEOREM

Definition : An expression having two terms is called a binomial. Thus $a+b$, $b+y$, $a+x$ are binomial expression.

1.4.1 Binomial Theorem for a positive integer :

When n is a positive integer,

$$(a+x)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}x + {}^nC_2\, a^{n-2}x^2 + \dots\dots + {}^nC_r\, a^{n-r}x^r + \dots + {}^nC_n\, x^n$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \dots$$

$$+ \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}a^{n-r}x^r + \dots\dots\dots + x^n$$

for all values of a and x.

Note : 1. The number of terms in the expansion $(a+x)^n$ is $n+1$.

2. The co-efficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... are called Binomial Co-efficient.

These are shortly written as $C_0, C_1, \dots\dots\dots C_n$

3. x^r occur in $(r+1)^{th}$ term and the co-efficient of x^r is ${}^nC_r$

4. $(r+1)^{th}$ term is $T_{r+1} = {}^nC_r\, a^{n-r}x^r$ It is called **general term**.

5. $(a-x)^n = (a+(-x))^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}(-x) + {}^nC_2 a^{n-2}(-x)^2 + \dots\dots + {}^nC_r\, a^{n-r}(-x)^r + \dots\dots\dots\dots + {}^nC_n(-x)^n$

$$= a^n - na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 - \dots\dots\dots$$

$$+(-1)^r \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}a^{n-r}x^r + \dots\dots + (-1)^n x^n$$

Example: In the expansion $(2+x)^9$ the total number of term is $9+1=10$.

4^{th} term is $T_4 = {}^9C_3\, 2^6 x^3$

8. In how many ways can the letters of the word SUNDAY be arranged? Ans: 720

9. How many words can be formed out of the letters ARTICLE, so that vowels always occupy the even places? Ans: 144

10. In how many ways can the letters of the word MISSISSIPPI be arranged? Ans: 34650

11. In how many ways can the letters of the word PERMUTATION be arranged keeping the vowels together? Ans: 302400

12. In how many ways can 7 examination papers be arranged so that the best and the worst papers never come together? Ans: $5! \times {}^6P_2$

13. In how many ways can 12 examination papers be arranged so that the best and the worst papers never come together? Ans: $10! \times {}^{11}P_2$

14. In how many ways 7 girls and 6 boys can sit at a round table so that no two boys are together? Ans: $6! \times {}^7P_6$

15. In how many ways 4 girls and 4 boys can sit at a round table so that no two girls are together? Ans: $3! \times {}^4P_4$

16. A class test is attended by 6 students sitting in a chair in a circle. In how many ways can they arrange themselves? Ans: 5!

17. How many different ways 2 boys can sit in 5 vacant seats? Ans: 20

18. In how many ways can the letters of the word COTTON be arranged so that the 2 T's do not come together? Ans: 120

19. In how many ways can 8 boys form a ring? Ans: 7!

20. In how many ways can 16 boys form a ring? Ans: 15!

21. How many numbers between 100 and 1000 can be formed with 2, 3, 4, 0, 8, 9 so that digits will not repeat? Ans: 100

22. How many 3 digit numbers less than 600 can be formed from 1, 2, 3, 4, 5, 9 so that digits may repeat? Ans: 180

23. How many numbers of not more than 5 digits can be formed with the digits 1, 2, 3? Ans: 363

24. Find the value of (i) ${}^{10}C_3$ (ii) ${}^{15}C_{15}$ (iii) ${}^{16}C_{15}$ (iv) ${}^{20}C_2$ (v) ${}^{30}C_{29}$ (vi) ${}^{11}C_5$

Ans: (i) 120 (ii) 1 (iii) 16 (iv) 190 (v) 30 (vi) 462

25. Find n (i) ${}^{2m}C_n = {}^{2m}C_{n+2}$ (ii) ${}^{15}C_n = {}^{15}C_{n-3}$ (iii) ${}^{12}C_{n+2} = {}^{12}C_{2n-5}$ (iv) ${}^nC_{15} = {}^nC_{10}$

Ans: (i) $m-1$ (ii) 9 (iii) 5 (iv) 25

26. (i) If ${}^nC_7 = {}^nC_{11}$ find ${}^{21}C_n$ (ii) If ${}^nC_{10} = {}^nC_3$ find ${}^nC_2$

(iii) If ${}^{20}C_r = {}^{20}C_{2r-1}$ find (a) ${}^{10}C_r$ (b) ${}^rC_5$

(iv) If ${}^nC_{16} = {}^nC_5$ find ${}^nC_3$ (v) If ${}^nC_2 = 28$ find ${}^nC_5$

Ans: (i) 1330 (ii) 78 (iii) a. 120 b. 21 (iv) 1330 (v) 56

27. Find n (i) ${}^{2n}C_4 : {}^nC_3 = 35:2$ (ii) ${}^nC_3 : {}^{n-1}C_3 = 4:3$

Ans: (i) 4 (ii) 12

28. If ${}^nP_r = 336$, ${}^nC_r = 56$ find n and r. Ans: $n=8$, $r=3$

29. How many groups each consisting of 6 students can be formed from 9 students? Ans: 84

30. A basket contains 10 mangoes. Find how many different selections one can make of 3 mangoes so as to include always a particular mango? Ans: 36

31. Out of 9 boys and 5 girls how many different groups can be formed, each consisting of 6 boys and 2 girls? Ans: 840

32. In how many ways on the occasion of birthday a girl can invite her one or more of 6 friends? Ans: 63

33. In how many ways can 10 boys be selected from 18 boys so as to include always 3 particular boys? Ans: 6435

34. In how many ways can 12 different toys be distributed equally among 3 children? Ans: 34650

35. In how many ways can 12 sweets be distributed equally among 4 boys? Ans: ${}^{12}C_3 \times {}^9C_3 \times {}^6C_3 \times 1$

36. From a group of 15 men in how many ways 9 man can be selected so that

(i) 3 particular men are always excluded?

(ii) 3 particular men are always included?

Ans: (i) 220 (ii) 924

Ex 35. In a group of 15 boys there are 7 boy scouts. In how many ways can 12 boys be selected so as to include (i) exactly 6 boy scouts, (ii) at least 6 boy scouts?

Soln: (i) 6 boy scouts come from group of 7 boy scouts in $^7C_6$ ways.

Other 6 boys will come from group of 8 boys in $^8C_6$ ways.

Required number of combination is $^7C_6 \times {}^8C_6 = 7 \times 28 = 196$

(ii) Number of boy scouts may be 6 or 7

Case 1. 6 boy scouts from group of 7 and 6 boys from group of 8.

Number of combination is $^7C_6 \times {}^8C_6 = 7 \times 28 = 196$

Case 2. 7 boy scouts from group of 7 and 5 boys from group of 8.

Number of combination is $^7C_7 \times {}^8C_5 = 1 \times 56 = 56$

Required number of combination is $196 + 56 = 252$

Ex 36. There are 6 points in a plane and A is one of them. If no three of them are collinear, find the number of triangles that can be drawn with A as vertex?

Soln: Any 3 points will form a triangle, since no three of them are collinear. Along with A we need to select 2 points from other 5 points. Number of combination is

$^5C_2 = 10$.

Ex 37. In how many ways 21 identical black balls and 19 identical red balls can be arranged so that no two red balls are together?

Soln: The red balls will have to be kept in 22 places i.e. in the space separated by black balls. Total number of arrangements is $^{22}C_{19} = 1540$.

Ex 38. If m parallel lines in a plane are intersected by another set of n parallel lines, find the number of parallelogram formed.

Soln: A parallelogram needs two pairs of parallel straight lines.

Number of parallelogram formed is $^mC_2 \times {}^nC_2$

Ex 39. **[2016]** There are 7 gentlemen and 3 ladies contesting for two vacancies. An elector can vote any number of candidates not exceeding the number of vacancies. In how many ways it is possible to vote?

Solution: Case I- 2 gentlemen are selected.

Number of combination- $^7C_2 = 21$

Case II- 2 ladies are selected. Number of combination- $^3C_2 = 3$

Case III- one gentlemen and one lady are selected.

Number of combination- $^7C_1 \times {}^3C_1 = 21$

Case IV- one lady is selected. Number of combination- $^3C_1 = 3$

Case V- one gentleman is selected. Number of combination- $^7C_1 = 7$

Thus total number of possibilities is- $21 + 3 + 21 + 3 + 7 = 55$.

Exercise 1.3

1. Find n if (i) $^nP_4 = 12 \times {}^nP_2$ (ii) $^nP_4 = 10 \times {}^{n-1}P_3$

(iii) $^nP_5 = 90 \times {}^{n-2}P_3$ (iv) $^8P_n = 1680$

(v) $^{n-2}P_3 = 10 \times {}^nP_2$ (vi) $^{n+2}P_3 = 5 \times {}^{n+1}P_2$

Ans: (i) 6 (ii) 10 (iii) 10 (iv) 4 (v) 3, 4 (vi) 3

2. Find n if (i) $^{n-1}P_3 : {}^{n+1}P_3 = 5 : 12$ (ii) $^nP_4 : {}^{n+1}P_4 = 5 : 9$

(iii) $^nP_5 : {}^nP_3 = 2 : 1$ (iv) $^{n+1}P_4 : {}^{n-1}P_3 = 72 : 5$

Ans: (i) 8 (ii) 8 (iii) 5 (iv) 8

3. (i) $^{n+r}P_2 = 90$, $^{n-r}P_2 = 30$ Find n and r.

(ii) $^{m+n}P_2 = 56$, $^{m-n}P_2 = 12$ Find m and n.

Ans: (i) $n = 8, r = 2$ (ii) $m = 6, n = 2$

4. $^nP_{r-1} : {}^nP_r : {}^nP_{r+1} = a : b : c$, show that $b^2 = a(b + c)$

5. In how many ways can the letters of the word COLLEGE be arranged? Ans: 1260

6. In how many ways 6 books out of 11 different books can be arranged in a shelf so that 3 particular books are always together? Ans: 8064

7. In how many ways can the letters of the word JALPAIGURI be arranged? Ans: 907200

$$=\frac{1}{2}\left(1+\frac{5}{8}x^2+\frac{75}{128}x^4+\frac{625}{1024}x^6+\frac{21875}{32768}x^8+\ldots\ldots\right)$$

(iii) $$(1-x^2)^{-2/3}=1+\left(-\frac{2}{3}\right)(-x^2)+\frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{2!}(-x^2)^2$$

$$+\frac{(-\frac{2}{3})(-\frac{2}{3}-1)(-\frac{2}{3}-2)}{3!}(-x^2)^3$$

$$+\frac{(-\frac{2}{3})(-\frac{2}{3}-1)(-\frac{2}{3}-2)(-\frac{2}{3}-3)}{4!}(-x^2)^4+\ldots\ldots$$

$$=1+\frac{2}{3}x^2+\frac{5}{9}x^4+\frac{40}{81}x^6+\frac{110}{240}x^8+\ldots\ldots$$

Ex 9. Write (i) 3rd term in $\frac{2+x}{(3-2x)^2}$ (ii) 4th term in $(1-3x)^{10/3}$

(iii) coefficient of x^{10} in $(1-2x^2)^{-5/2}$ (iv) coefficient of x^2 in $\frac{1+2x}{(1-3x)^3}$

Soln: (i) $$\frac{2+x}{(3-2x)^2}=(2+x)(3-2x)^{-2}=(2+x)\{3^{-2}(1-\frac{2}{3}x)^{-2}\}$$

$$=3^{-2}(2+x)\{1+(-2)(-\frac{2}{3}x)+\frac{(-2)(-2-1)}{2!}(-\frac{2}{3}x)^2+\ldots\ldots$$

$$=3^{-2}(2+x)\{1+\frac{4}{3}x+\frac{4}{3}x^2+\ldots\ldots\}$$

$$=\frac{1}{9}\{2+(1+\frac{8}{3})x+(\frac{4}{3}+\frac{8}{3})x^2+\ldots\ldots\}$$

$$=\frac{1}{9}(2+\frac{11}{3}x+4x^2+\ldots\ldots)$$

Here 3rd term is $\frac{4}{9}x^2$

(ii) $$(1-3x)^{10/3}=1+\frac{10}{3}(-3x)+\frac{\frac{10}{3}(\frac{10}{3}-1)}{2!}(-3x)^2$$

$$+\frac{\frac{10}{3}(\frac{10}{3}-1)(\frac{10}{3}-2)}{3!}(-3x)^3+\ldots\ldots$$

Here 4th term is $$\frac{\frac{10}{3}(\frac{10}{3}-1)(\frac{10}{3}-2)}{3!}(-3x)^3=-\frac{10\times7\times4\times27}{3\times3\times3\times6}x^3=-\frac{140}{3}x^3$$

(iii) The term containing x^{10} in $(1-2x^2)^{-5/2}$ is 6th term.

$$T_6=\frac{(-\frac{5}{2})(-\frac{5}{2}-1)(-\frac{5}{2}-2)(-\frac{5}{2}-3)(-\frac{5}{2}-4)}{5!}(-2x^2)^5$$

$$=\frac{5\times7\times9\times11\times13\times32}{32\times120}x^{10}=\frac{3003}{8}x^{10}$$

Coefficient of x^{10} is $\frac{3003}{8}$

(iv) $$\frac{1+2x}{(1-3x)^3}=(1+2x)(1-3x)^{-3}$$

$$=(1+2x)\{1+(-3)(-3x)+\frac{(-3)(-3-1)}{2!}(-3x)^2+\ldots\ldots\}$$

$$=(1+2x)(1+9x+54x^2+\ldots\ldots)$$

The term containing x^2 is $2x\times9x+54x^2=72x^2$

Coefficient of x^2 is 72

(v) The general term is $T_{r+1} = {}^{12}C_r\,(2x^3)^{12-r}(-\frac{1}{x})^r$

$$= (-1)^r\, {}^{12}C_r \times 2^{12-r} \times x^{24-3r}$$

T_{r+1} independent of x if $24-3r=0 \quad \Rightarrow r=8$

The required term is $T_9 = (-1)^8 \times {}^{12}C_8 \times 2^{12-8} = 495 \times 16 = 7920$

Ex 7. Apply binomial theorem to find the value of:

(i) $(1+\sqrt{3})^7+(1-\sqrt{3})^7$ (ii) $(x+\sqrt{1-x^2})^4+(x-\sqrt{1-x^2})^4$

(iii) $(2\sqrt{5}+1)^5-(2\sqrt{5}-1)^5$ (iv) $(2+\sqrt{19})^3+(2-\sqrt{19})^3$

(v) **[QP 2016]** $(1+\sqrt{7})^5+(1-\sqrt{7})^5$

Soln: (i) $(1+\sqrt{3})^7+(1-\sqrt{3})^7 = \{1+{}^7C_1\sqrt{3}+{}^7C_2(\sqrt{3})^2+{}^7C_3(\sqrt{3})^3+{}^7C_4(\sqrt{3})^4$
$+{}^7C_5(\sqrt{3})^5+{}^7C_6(\sqrt{3})^6+(\sqrt{3})^7\}+\{1-{}^7C_1\sqrt{3}+{}^7C_2(\sqrt{3})^2$
$-{}^7C_3(\sqrt{3})^3+{}^7C_4(\sqrt{3})^4-{}^7C_5(\sqrt{3})^5+{}^7C_6(\sqrt{3})^6-(\sqrt{3})^7\}$
$=2\{1+{}^7C_2(\sqrt{3})^2+{}^7C_4(\sqrt{3})^4+{}^7C_6(\sqrt{3})^6\}$
$=2(1+63+315+189)=1136$

(ii) $(x+\sqrt{1-x^2})^4+(x-\sqrt{1-x^2})^4 = \{x^4+{}^4C_1x^3\sqrt{1-x^2}+{}^4C_2x^2(\sqrt{1-x^2})^2$
$+{}^4C_3x(\sqrt{1-x^2})^3+(\sqrt{1-x^2})^4\}+\{x^4-{}^4C_1x^3\sqrt{1-x^2}$
$+{}^4C_2x^2(\sqrt{1-x^2})^2-{}^4C_3x(\sqrt{1-x^2})^3+(\sqrt{1-x^2})^4\}$
$=2\{x^4+{}^4C_2x^2(\sqrt{1-x^2})^2+(\sqrt{1-x^2})^4\}$
$=2\{x^4+6x^2(1-x^2)+1-2x^2+x^4\}=2(1+4x^2-4x^4)$

(iii) $(2\sqrt{5}+1)^5-(2\sqrt{5}-1)^5=\{(2\sqrt{5})^5+{}^5C_1(2\sqrt{5})^4+{}^5C_2(2\sqrt{5})^3$
$+{}^5C_3(2\sqrt{5})^2+{}^5C_4(2\sqrt{5})+1\}-\{(2\sqrt{5})^5-{}^5C_1(2\sqrt{5})^4$
$+{}^5C_2(2\sqrt{5})^3-{}^5C_3(2\sqrt{5})^2+{}^5C_4(2\sqrt{5})-1\}$

$=2\{{}^5C_1(2\sqrt{5})^4+{}^5C_3(2\sqrt{5})^2+1\}$
$=2(5\times16\times25+10\times20+1)=4402$

(iv) $(2+\sqrt{19})^3+(2-\sqrt{19})^3=\{2^3+{}^3C_1\times2^2(\sqrt{19})+{}^3C_2\times2(\sqrt{19})^2$
$+(\sqrt{19})^3\}+\{2^3-{}^3C_1\times2^2(\sqrt{19})+{}^3C_2\times2(\sqrt{19})^2-(\sqrt{19})^3\}$
$=2\{2^3+{}^3C_2\times2(\sqrt{19})^2\}=244$

(v) $(1+\sqrt{7})^5+(1-\sqrt{7})^5=2\{1+{}^5C_2(\sqrt{7})^2+{}^5C_4(\sqrt{7})^4\}=2(1+70+245)$
$=632$

Ex 8. Expand: (i) $(1+2x)^{-4}$ (ii) $\frac{1}{(4-5x^2)^{1/2}}$ (iii) $(1-x^2)^{-2/3}$

Soln: (i) $(1+2x)^{-4}=1+(-4)(2x)+\frac{(-4)(-4-1)}{2!}(2x)^2+\frac{(-4)(-4-1)(-4-2)}{3!}(2x)^3$
$+\frac{(-4)(-4-1)(-4-2)(-4-3)}{4!}(2x)^4+\ldots\ldots$
$=1-8x+40x^2-160x^3+560x^4+\ldots\ldots$

(we assume $|2x|<1$)

(ii) $\frac{1}{(4-5x^2)^{1/2}}=(4-5x^2)^{-1/2}=4^{-1/2}(1-\frac{5x^2}{4})^{-1/2}$ (we assume $\left|\frac{5x^2}{4}\right|<1$)

$$=\frac{1}{2}\{1+(-\frac{1}{2})(-\frac{5x^2}{4})+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-\frac{5x^2}{4})^2$$
$$+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(-\frac{5x^2}{4})^3$$
$$+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{4!}(-\frac{5x^2}{4})^4+\ldots\ldots\}$$

Soln: (i) The general term is $T_{r+1} = {}^{10}C_r\, x^{10-r}(-x^2)^r = (-1)^r\, {}^{10}C_r\, x^{10+r}$

T_{r+1} contains x^{17} if $10+r=17 \quad \Rightarrow r=7$

The required term is $T_8 = (-1)^7 \times {}^{10}C_7\, x^{10+7} = -\,{}^{10}C_7\, x^{17}$

Thus coefficient of x^{17} in $(x-x^2)^{10}$ is $-\,{}^{10}C_7 = -120$

(ii) The general term is $T_{r+1} = {}^{20}C_r (2x^2)^{20-r}(\frac{1}{x})^r = {}^{20}C_r \times 2^{20-r} \times x^{40-3r}$

T_{r+1} contains x^{-2} if $40-3r=-2 \quad \Rightarrow r=14$

The required term is $T_{15} = {}^{20}C_{14} \times 2^6 x^{-2}$

Thus coefficient of x^{-2} in $(2x^2+\frac{1}{x})^{20}$ is ${}^{20}C_{14} \times 2^6$

(iii) The general term is $T_{r+1} = {}^{12}C_r (x^2)^{12-r}(-\frac{1}{x^3})^r = (-1)^r\, {}^{12}C_r\, x^{24-5r}$

T_{r+1} contains x^{-11} if $24-5r=-11 \quad \Rightarrow r=7$

The required term is $T_8 = (-1)^7 \times {}^{12}C_7\, x^{-11} = -\,{}^{12}C_7\, x^{-11}$

Thus coefficient of x^{-11} in $(x^2-\frac{1}{x^3})^{12}$ is $-\,{}^{12}C_7 = = -792$

(iv) The general term is $T_{r+1} = {}^8C_r (\frac{3}{x^2})^{8-r}(-\frac{x^3}{2})^r$

$$= (-1)^r\, {}^8C_r \times 3^{8-r} \times (\frac{1}{2})^r x^{5r-16}$$

T_{r+1} contains x^{-1} if $5r-16=-1 \quad \Rightarrow r=3$

The required term is $T_4 = (-1)^3 \times {}^8C_3 \times 3^5\, 2^{-3} \times x^{-1} = -7 \times 3^5 x^{-1}$

Thus coefficient of x^{-1} in $(\frac{3}{x^2}-\frac{x^3}{2})^8$ is -7×3^5

Ex 6. Find the term independent of x in (i) **[QP 2009, 2016]** $(\frac{3x^2}{2}-\frac{1}{3x})^9$

(ii) **[QP 2009]** $(2x+\frac{1}{3x^2})^9$ (iii) **[QP 2015]** $(2x^4-\frac{1}{x})^{10}$

(iv) **[QP 2011]** $(x+\frac{1}{x})^{19}$ (v) **[QP 2009, 2015, 2016]** $(2x^2-\frac{1}{x})^{12}$

Soln: (i) The general term is $T_{r+1} = {}^9C_r (\frac{3x^2}{2})^{9-r}(-\frac{1}{3x})^r$

$$= (-1)^r \times 3^{9-2r} \times (\frac{1}{2})^{9-r} x^{18-3r}$$

T_{r+1} independent of x if $18-3r=0 \quad \Rightarrow r=6$

The required term is $T_7 = (-1)^6 \times {}^9C_6 \times 3^{-3} \times 2^{-3} = 84 \times 6^{-3}$

(ii) The general term is $T_{r+1} = {}^9C_r\, (2x)^{9-r}(\frac{1}{3x^2})^r$

$$= {}^9C_r \times 2^{9-r} \times (\frac{1}{3})^r x^{9-3r}$$

T_{r+1} independent of x if $9-3r=0 \quad \Rightarrow r=3$

The required term is $T_4 = {}^9C_3 \times 2^6\, 3^{-3} = 84 \times 64 \times 3^{-3}$

(iii) The general term is $T_{r+1} = {}^{10}C_r (2x^4)^{10-r}(-\frac{1}{x})^r$

$$= (-1)^r\, {}^{10}C_r \times 2^{10-r} \times x^{40-5r}$$

T_{r+1} independent of x if $40-5r=0 \quad \Rightarrow r=8$

The required term is $T_9 = (-1)^8 \times {}^{10}C_8 \times 2^{10-8} = 45 \times 4 = 180$

(iv) The general term is $T_{r+1} = {}^{19}C_r\, x^{19-r}(\frac{1}{x})^r$

$$= {}^{19}C_r\, x^{19-2r}$$

T_{r+1} independent of x if $19-2r=0 \quad \Rightarrow r=9.5$ an impossibility.

There is no term independent of x in the expansion.

$$+{}^6C_3(4a)^3(-3b^2c)^3+{}^6C_4(4a)^2(-3b^2c)^4+{}^6C_5(4a)(-3b^2c)^5$$
$$+{}^6C_6(-3b^2c)^6$$
$$=4^6a^6-6\times4^5\times3a^5b^2c+15\times4^4\times9a^4b^4c^2-20\times64\times27a^3b^6c^3$$
$$+15\times16\times81a^2b^8c^4-6\times4\times3^5\,ab^{10}c^5+3^6\,b^{12}c^6$$

Ex 2. Find the 10^{th} term from beginning and the 10^{th} term from end of the expansion $(2x+\frac{1}{x^2})^{25}$

Soln: We have $(r+1)^{th}$ term $T_{r+1}={}^nC_r\,a^{n-r}x^r$

10^{th} term from the begining i.e. $(9+1)^{th}$ term of $(2x+\frac{1}{x^2})^{25}$ is

$$T_{9+1}={}^{25}C_9\,(2x)^{25-9}(\frac{1}{x^2})^9={}^{25}C_9 2^{16}\frac{1}{x^2}$$

Again 10^{th} term of from the end is $=T_{26-9}=T_{17}$

17^{th} i.e $(16+1)^{th}$ term of $(2x+\frac{1}{x^2})^{25}$ is

$$T_{16+1}={}^{25}C_{16}\,(2x)^{25-16}(\frac{1}{x^2})^{16}={}^{25}C_{16}\,2^9\,\frac{1}{x^{23}}$$

Ex 3. Find (i) 5^{th} term of the expansion $(x-\frac{1}{2x})^7$

(ii) **[QP 2010]** 8^{th} term of the expansion $(1+\frac{1}{x})^{17}$

(iii) **[QP 2013]** 12^{th} term of the expansion $(3x^2-\frac{1}{2x})^{21}$

(iv) **[QP 2015]** 12^{th} term in $(2x+\frac{1}{3x})^{19}$

Soln: (i) $T_5=T_{4+1}={}^7C_4\,x^{7-4}(-\frac{1}{2x})^4=35\times\frac{1}{16x}$

(ii) $T_8=T_{7+1}={}^{17}C_7(\frac{1}{x})^7={}^{17}C_7\,\frac{1}{x^7}$

(iii) $T_{12}=T_{11+1}={}^{21}C_{11}\,(3x^2)^{21-11}(-\frac{1}{2x})^{11}=-{}^{21}C_{11}\times3^{10}\times2^{-11}x^9$

(iv) $T_{12}={}^{19}C_{11}(2x)^8(\frac{1}{3x})^{11}={}^{19}C_{11}\times2^8\times3^{-11}x^{-3}$

Ex 4. Find the middle terms of (i) **[QP 2011]** $(3x-\frac{x^3}{6})^9$

(ii) **[QP 2009]** $(2x+\frac{1}{x^2})^{18}$ (iii) **[QP 2012]** $(x+\frac{2}{x^2})^{17}$

Soln: (i) There are 10 terms. Middle terms are T_5 and T_6

$$T_5=T_{4+1}={}^9C_4\,(3x)^{9-4}(-\frac{x^3}{6})^4=126\times\frac{3^5}{6^4}\,x^{17}=126\times\frac{3}{16}x^{17}$$

$$T_6=T_{5+1}={}^9C_5\,(3x)^{9-5}(-\frac{x^3}{6})^5=-126\times\frac{3^4}{6^5}x^{19}=-126\times\frac{1}{96}\,x^{19}$$

(ii) There are 19 terms. Middle term is T_{10}

$$T_{10}=T_{9+1}={}^{18}C_9\,(2x)^{18-9}(\frac{1}{x^2})^9={}^{18}C_9\times2^9\times\frac{1}{x^9}$$

(iii) There are 18 terms. Middle terms are T_9 and T_{10}

$$T_9=T_{8+1}={}^{17}C_8\,x^{17-8}(\frac{2}{x^2})^8={}^{17}C_8\times2^8\times\frac{1}{x^7}$$

$$T_{10}=T_{9+1}={}^{17}C_9\,x^{17-9}(\frac{2}{x^2})^9={}^{17}C_9\times2^9\times\frac{1}{x^{10}}$$

Ex 5. Find the coefficient of

(i) **[QP 2010, 2014]** x^{17} in $(x-x^2)^{10}$ (ii) **[QP 2011]** x^{-2} in $(2x^2+\frac{1}{x})^{20}$

(iii) **[QP 2013]** x^{-11} in $(x^2-\frac{1}{x^3})^{12}$ (iv) **[QP 2013]** x^{-1} in $(\frac{3}{x^2}-\frac{x^3}{2})^9$

Now $a^x \times a^y = a^{x+y} = m \times n$

$\Rightarrow \log_a (m \times n) = x + y = \log_a m + \log_a n$ Proved.

Law 2: $\log_a (\frac{m}{n}) = \log_a m - \log_a n \ (m, n > 0)$

Proof: Let $\log_a m = x \Rightarrow a^x = m$ and $\log_a n = y \Rightarrow a^y = n$

Now $\frac{a^x}{a^y} = a^{x-y} = \frac{m}{n}$

$\Rightarrow \log_a (\frac{m}{n}) = x - y = \log_a m - \log_a n$ Proved.

Law 3: $\log_a (m^n) = n \log_a m$

Proof: Let $\log_a m = x \Rightarrow a^x = m$

Now $(a^x)^n = m^n = a^{nx}$

$\Rightarrow \log_a (m^n) = nx = n \log_a m$ Proved.

1.5.3 Special cases in logarithm:

Case1. Since $a^0 = 1$ therefore $\log_a 1 = 0$ for any a

Thus $\log_5 1 = 0$, $\log_9 1 = 0$ etc.

Case 2. Since $a^1 = a$ therefore $\log_a a = 1$ for any a

Thus $\log_5 5 = 1$, $\log_9 9 = 1$

Case 3. Since $a^{-1} = \frac{1}{a}$ therefore $\log_a (\frac{1}{a}) = -1$

Thus $\log_5 (\frac{1}{5}) = -1$, $\log_9 (\frac{1}{9}) = -1$

1.5.4 Properties of logarithm : (Change of base)

$\log_a m = \log_b m \times \log_a b$

Proof: Let $x = \log_a m$, $y = \log_b m$, and $z = \log_a b$

$\Rightarrow a^x = m,\ b^y = m,\ a^z = b$

We see $a^x = b^y = m$

or, $a^x = b^y = (a^z)^y = a^{zy} \quad \Rightarrow a^x = a^{zy}$

$\Rightarrow x = yz \quad \Rightarrow \log_a m = \log_b m \times \log_a b$ Proved.

Cor 1. $\log_a b \times \log_b a = 1$

Proof: $\log_a b \times \log_b a = \log_a a = 1$

Cor 2. $\log_a m = \log_b m / \log_b a$

Worked out Examples:

Ex 1. Find the logarithm of: (i) 125 to the base $\sqrt{5}$ (ii) 784 to the base $2\sqrt{7}$

Soln: (i) We know that $a^x = N \Rightarrow \log_a N = x$

Thus $(\sqrt{5})^x = 125 \Rightarrow \log_{\sqrt{5}} 125 = x$

Since $(\sqrt{5})^6 = 125 \Rightarrow \log_{\sqrt{5}} 125 = 6$

(ii) Here we see $(2\sqrt{7})^x = 784 = (2\sqrt{7})^4$

$\Rightarrow \log_{2\sqrt{7}} 784 = 4$

Ex 2. Find the base a of the logarithm if

(i) **[QP 2017]** $\log_a 1728 = 6$ (ii) $\log_a 16 = -4$

(iii) **[QP 2013]** $\log_a 324 = 4$

Soln: (i) $\log_a 1728 = 6 \quad \Rightarrow a^6 = 1728 = 12^3 = (4 \times 3)^3$

$= (2\sqrt{3})^6$

$\Rightarrow a = 2\sqrt{3}$ We have $\log_{2\sqrt{3}} 1728 = 6$

(ii) $\log_a 16 = -4 \quad \Rightarrow a^{-4} = 16 \quad \Rightarrow \frac{1}{a^4} = 16 \quad \Rightarrow a^4 = \frac{1}{16} = \frac{1}{2^4} = (\frac{1}{2})^4$

$\Rightarrow a = \frac{1}{2}$

(iii) $\log_a 324 = 4 \quad \Rightarrow a^4 = 324 = 18^2 = (9 \times 2)^2 = (3\sqrt{2})^4$

$\Rightarrow a = 3\sqrt{2}$ We have $\log_{3\sqrt{2}} 324 = 4$

Ex 3. Find the value of: (i) $\log_6 216$ (ii) **[QP 2015]** $\log_3 \log_2 16$

(iii) **[QP 2015]** $\log_2 \log_3 \log_2 512$ (iv) **[QP 2009]** $\log_2 (\sqrt{6}) + \log_2 (\sqrt{\frac{2}{3}})$

Soln: (i) $\log_6 216 = \log_6 (6)^3 = 3 \log_6 6 = 3 \times 1 = 3$

6. Find the constant term of $(x-\frac{1}{x})^4$ Ans: $T_3=6$

7. In the expansion of $(1+x)^{25}$ the coefficient of $(2r+1)^{th}$ and $(r+5)^{th}$ terms are equal. Find r. Ans: $r=7$

8. Apply binomial theorem to find the value of:

(i) $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$ (ii) $(\sqrt{3}+1)^5-(\sqrt{3}-1)^5$

(iii) $(5+2\sqrt{7})^3+(5-2\sqrt{7})^3$ (iv) $(1+3\sqrt{3})^4+(1-3\sqrt{3})^4$

Ans: (i) 82 (ii) 152 (iii) 1090 (iv) 1784

9. Expand upto 4^{th} term: (i) $(1+2x)^{-3}$ (ii) $\sqrt{4+3x}$ (iii) $\frac{1}{\sqrt[3]{2-x^3}}$

Ans: (i) $1-6x+24x^2-80x^3+\ldots\ldots$ (ii) $2(1+\frac{3}{8}x-\frac{9}{128}x^2+\frac{3^3}{4^5}x^3-\ldots\ldots\ldots)$

(iii) $2^{-1/3}(1+\frac{x^3}{6}+\frac{x^6}{18}+\frac{7x^9}{12}+\ldots\ldots\ldots)$

10. Write coefficient of: (i) x^8 in $\sqrt{9+x^2}$ (ii) x^4 in $\frac{1}{\sqrt[3]{3-x}}$ (iii) x^3 in $\frac{\sqrt{x^2+1}}{1-x}$

Ans: (i) $\frac{-5}{2^7 3^7}$ (ii) $3^{-\frac{1}{3}}\left(\frac{44}{3^4 5^4}\right)$ (iii) $\frac{3}{2}$

11. Apply binomial theorem to evaluate upto 5 decimal places:

(i) $\sqrt{99}$ (ii) 98^4 (iii) $\sqrt[3]{998}$ (iv) $(1.01)^5$ (v) $\sqrt[3]{1001}$

Ans: (i) 9.944988 (ii) 92236816 (iii) 9.99334 (iv) 1.05101 (v) 10.00333

12. If $C_0, C_1, C_2, \ldots\ldots\ldots\ldots, C_n$ are binomial coefficient corresponding to the power n prove that

(i) $C_1-2C_2+3C_3-\ldots\ldots\ldots\ldots+(-1)^{n-1}nC_n=0$

(ii) $C_0C_n+C_1C_{n-1}+C_2C_{n-2}+\ldots\ldots\ldots\ldots\ldots\ldots+C_nC_0=\frac{(2n)!}{n!n!}$

(iii) $C_0+3C_1+5C_2+7C_3\ldots\ldots\ldots\ldots+(2n+1)C_n=(n+1)2^n$

1.5 LOGARITHM

John Napier was the inventor of Logarithm. For the numerical calculus, Logarithm is very important. With the help of Logarithm we can do multiplication, computations of power and roots, etc. There are three systems of Logarithm.

1. Common Logarithm
2. Natural Logarithm (or Napier Logarithm)
3. Binary Logarithm

Common Logarithm: It was first introduced by Prof. Henry Briggs. It is a kind of Logarithm developed with respect to the base 10 of decimal system. It is used for numerical computation.

Natural Logarithm: It was first introduced by John Napier (1550-1617). It is a kind of Logarithm developed with respect to the base e where $2<e<3$ ($e=2.7183$). It is used for theoretical purpose.

Binary Logarithm: Its base is 2 and is used in Computer Science.

1.5.1 Definition of Logarithm: Let us consider the equation $a^x=N(a>0, a\neq 1)$, where a is called base and x is the index or the power.
Now x is called Logarithm of N to the base a and is written as $x=\log_a N$. This is read as Logarithm of N to the base a.

Examples: (i) $3^2=9 \Rightarrow 2=\log_3 9$.

(ii) $3^{-2}=\frac{1}{9} \Rightarrow -2=\log_3\frac{1}{9}$

(iii) $3^4=81 \Rightarrow 4=\log_3 81$

1.5.2 Laws of Logarithm:

Law 1: $\log_a(m\times n)=\log_a m+\log_a n$ $(m, n>0)$

Proof: Let $\log_a m=x \Rightarrow a^x=m$ and $\log_a n=y \Rightarrow a^y=n$

$$= ({}^nC_0 \times {}^nC_0) + ({}^nC_0 \times {}^nC_1 + {}^nC_1 \times {}^nC_0)\, x$$
$$+ ({}^nC_0 \times {}^nC_2 + {}^nC_1 \times {}^nC_1 + {}^nC_2 \times {}^nC_0)\, x^2 + \ldots\ldots$$
$$+ {}^nC_n \times {}^nC_n\, x^{2n}$$

$$= C_0 C_0 + (C_0 C_1 + C_1 C_0) x + (C_0 C_2 + C_1 C_1 + C_2 C_0) x^2 + \ldots\ldots$$
$$+ (C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \ldots\ldots + C_n C_0) x^n$$
$$+ \ldots\ldots + C_n C_n x^{2n}$$

We have now

$$C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \ldots\ldots + C_n C_0 = {}^{2n}C_n = \frac{(2n)!}{n!\,n!}$$

Since $C_r = C_{n-r}$ hence

$$C_0 C_n = C_0^{\,2},\ C_1 C_{n-1} = C_1^{\,2}, \ldots\ldots, C_n C_0 = C_n^{\,2}$$

$$\Rightarrow C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots + C_n^{\,2} = \frac{(2n)!}{n!\,n!}$$

Exercise 1.4

1. Expand the following: (i) $(x + \frac{1}{x})^4$ (ii) $(x - \frac{1}{2x})^5$ (iii) $(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}})^6$

(iv) $(x - 1)^6$ (v) $(2x + 3y)^4$

Ans: (i) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$ (ii) $x^5 - \frac{5x^3}{2} + \frac{5x}{2} - \frac{5}{4x} + \frac{5}{16x^3} - \frac{1}{32x^5}$

(iii) $\frac{x^3}{a^3} - 6\frac{x^2}{a^2} + 15\frac{x}{a} - 20 + 15\frac{a}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$

(iv) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

(v) $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$

2. Find the (i) 5^{th} term in $(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}})^6$ (ii) 3^{rd} term in $(1 - \frac{x}{2})^7$

(iii) 4^{th} term in $(\frac{x}{a} - \frac{a}{x})^{10}$ (iv) general term in $(x + \frac{1}{x})^n$

(v) 11^{th} term in $(4x - \frac{1}{2\sqrt{x}})^{15}$ (vi) 5^{th} term in $(x - \frac{1}{2x})^9$

Ans: (i) $15\frac{a}{x}$ (ii) $\frac{21}{4}x^2$ (iii) $-120\frac{x^4}{a^4}$ (iv) ${}^nC_r\, x^{n-2r}$ (v) 3003 (vi) $\frac{5}{16x^3}$

3. Find the coefficient of (i) x^4 in $(x^4 + \frac{1}{x^3})^{15}$ (ii) x^{-1} in $(2x^2 - \frac{1}{x})^{10}$

(iii) x^7 in $(x - x^2)^{10}$ (iv) x^{32} in $(x^4 - \frac{1}{x^3})^{15}$ (v) x^{10} in $(2x^2 - \frac{3}{x})^{11}$

Ans: (i) 6435 (ii) -960 (iii) 0 (iv) 1365 (v) 3421440

4. Find the term independent of x: (i) $(x + \frac{1}{x})^{10}$ (ii) $(2x + \frac{1}{\sqrt{x}})^{15}$ (iii) $(x - \frac{1}{x^2})^{12}$

(iv) $(3x - \frac{1}{x^3})^8$ (v) $(9x^2 - \frac{1}{3x})^{12}$ (vi) $(x^2 + \frac{1}{x})^{12}$ (vii) $(x + \frac{1}{x})^{2n}$

Ans: (i) 252 (ii) 96096 (iii) 495 (iv) 20412 (v) 495 (vi) 495 (vii) $\frac{(2n)!}{n!\,n!}$

5. Find the middle terms of: (i) $(\frac{x}{a} + \frac{a}{x})^{20}$ (ii) $(a + x)^{21}$ (iii) $(x + \frac{1}{x})^7$

(iv) $(x + \frac{1}{x})^8$ (v) $(\frac{x}{y} - \frac{y}{x})^7$ (vi) $(2x + \frac{1}{3x^2})^9$

Ans: (i) ${}^{20}C_{10}$ (ii) ${}^{21}C_{10}\, a^{11} x^{10},\ {}^{21}C_{11}\, a^{10} x^{11}$ (iii) $35x, \frac{35}{x}$

(iv) 70 (v) $-\frac{35x}{y}, \frac{35y}{x}$ (vi) $\frac{448}{9x^3}, \frac{224}{27x^6}$

Ex 10. Apply binomial theorem to evaluate up to 3 decimal places:

(i) $(.999)^5$ (ii) $\sqrt[3]{999}$ (iii) $\sqrt[4]{16.08}$

Soln: (i) $(.999)^5 = (1-.001)^5 = 1 - {}^5C_1(.001) + {}^5C_2(.001)^2 - {}^5C_3(.001)^3 + {}^5C_4(.001)^4 - (.001)^5$

$= 1 - .005 + 10^{-5} - 10^{-8} + 5\times10^{-12} - 10^{-15}$

$= .995$

(ii) $\sqrt[3]{999} = (1000-1)^{1/3} = (1000)^{1/3}(1-.001)^{1/3}$

$$= 10\{1 + \frac{1}{3}(-.001) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(-.001)^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(-.001)^3 + \ldots\ldots\ldots\ldots\}$$

$= 10\{1 - .000333 - .00000011 + \ldots\ldots\}$

$= 10 \times .99966689 = 9.997$

(iii) $\sqrt[4]{16.08} = (16+.08)^{1/4} = 16^{1/4}(1+.005)^{1/4}$

$$= 2\{1 + \frac{1}{4}(.005) + \frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}(.005)^2 + \frac{\frac{1}{4}(\frac{1}{4}-1)(\frac{1}{4}-2)}{3!}(.005)^3 + \ldots\ldots\ldots\ldots\}$$

$= 2\{1 + .00125 - .00000234 + \ldots\ldots\} = 2.00249532 = 2.002$

Ex 11. If $C_0, C_1, C_2, \ldots\ldots\ldots, C_n$ are binomial coefficient corresponding to the power n prove that

(i) $C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_n = 2^n$

(ii) **[QP 2015]** $C_0 + C_2 + C_4 + \ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{n-1}$

(iii) $C_1 + C_2 + C_3 + \ldots\ldots\ldots + C_n = 1 + 2 + 2^3 + \ldots\ldots\ldots\ldots + 2^{n-1}$

(iv) **[QP 2014]** $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots + nC_n = n2^{n-1}$

(v) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots + C_n^2 = \frac{(2n)!}{n!n!}$

Soln: (i) LHS $= C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_n$

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n$

$= {}^nC_0\,1^n + {}^nC_1\,1^{n-1}\times1 + {}^nC_2\,1^{n-2}\times1^2 + \ldots\ldots\ldots + {}^nC_n\,1^n$

$= (1+1)^n = 2^n =$ RHS

(ii) $C_0 - C_1 + C_2 - \ldots\ldots\ldots\ldots + (-1)^n C_n$

$= {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots\ldots\ldots + (-1)^n\,{}^nC_n$

$= {}^nC_0\,1^n + {}^nC_1\,1^{n-1}\times(-1) + {}^nC_2\,1^{n-2}\times(-1)^2 + \ldots\ldots\ldots\ldots + {}^nC_n(-1)^n$

$= (1-1)^n = 0$

Hence $C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots\ldots$

$= \frac{1}{2}\times2(C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots) = \frac{1}{2}\times2(C_1 + C_3 + C_5 + \ldots\ldots\ldots\ldots)$

$= \frac{1}{2}\times(C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots + C_n) = \frac{1}{2}\times2^n = 2^{n-1}$

(iii) $C_1 + C_2 + C_3 + \ldots\ldots\ldots\ldots + C_n = -C_0 + C_0 + C_1 + C_2 + C_3 + \ldots\ldots\ldots\ldots + C_n$

$= -1 + 2^n = 2^n - 1 = \frac{2^n-1}{2-1} = 1 + 2 + 2^3 + \ldots\ldots\ldots\ldots + 2^{n-1}$

(iv) $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots\ldots + nC_n$

$$= n + 2\times\frac{n!}{2!(n-2)!} + 3\times\frac{n!}{3!(n-3)!} + \ldots\ldots\ldots\ldots + n\times\frac{n!}{n!(n-n)!}$$

$$= n\left(\frac{(n-1)!}{0!(n-1)!} + \frac{(n-1)!}{1!(n-2)!} + \frac{(n-1)!}{2!(n-3)!} + \ldots\ldots\ldots\ldots + \frac{(n-1)!}{(n-1)!0!}\right)$$

$= n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots\ldots\ldots + {}^{n-1}C_{n-1})$

$= n\times2^{n-1} = n2^{n-1}$

(v) $(1+x)^{2n} = (1+x)^n(1+x)^n$

We equate coefficient of x^n

$(1+x)^{2n} = 1 + {}^{2n}C_1x + {}^{2n}C_2x^2 + {}^{2n}C_3x^3 + \ldots\ldots\ldots\ldots + {}^{2n}C_nx^n + \ldots\ldots\ldots\ldots + {}^{2n}C_{2n}x^{2n}$

$(1+x)^n(1+x)^n = ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_nx^n)$
$\times({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_nx^n)$

10. Solve: (i) $\log_x 3 + \log_x 9 + \log_x 729 = 9$ Ans: 3

(ii) $\log_{10}(3x-2) - \log_{10}(x-1) = 1$ Ans: $\frac{8}{7}$

11. If $\log(x^3y^2) = 9$, $\log\left(\frac{x}{y}\right) = 2$ then prove that $\log x = 3$ and $\log y = 1$

12. If $\log_x b = 10$, $\log_{8x}(32b) = 5$ then find x. Ans: 3

13. If $a^2 + b^2 = 14ab$ then prove that $\log\frac{a+b}{4} = \frac{1}{2}(\log a + \log b)$

14. If $a^2 + b^2 = 6ab$ then prove that $2\log(a+b) = \log a + \log b + 3\log 2$

15. If $a^2 + b^2 = 11ab$ then prove that $\log\frac{a-b}{3} = \frac{1}{2}(\log a + \log b)$

16. If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$ then prove that $x^a y^b z^c = 1$

17. If $\log_a(ab) = x$, then prove that $\log_b(ab) = \frac{x}{x-1}$

18. Find the value of a if $\log_a b = 6$, $\log_{14a}(8b) = 3$. Ans: 7

19. Prove that $(yz)^{\log y - \log z} \times (zx)^{\log z - \log x} \times (xy)^{\log x - \log y} = 1$

20. Find the value of $\log_a x$ if $x = \sqrt{a\sqrt{a\sqrt{a\ldots\ldots}}}$ Ans: 1

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1.6 SERIES

1.6.1 Arithmetic Progression : A progression $a_1, a_2, a_3, \ldots\ldots$ is called Arithmetic Progression (in short AP) if

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots\ldots\ldots\ldots = a_{n+1} - a_n = \text{constant}, \ \forall\, n \in \mathbb{N}$$

The constant quantity is called common difference (c.d.) and it is denoted by d.

[The sequence whose terms progress (either in increasing or in decreasing order) in a certain pattern is called a **Progression.**]

common difference = any term – its preceding term

Note: The general representation of AP is

$$a,\ a+d,\ a+2d,\ a+3d, \ldots\ldots\ldots\ldots$$

Where a is 1st term or $t_1 = a$, d is common difference, and $t_n = a + (n-1)d$

Examples of AP series

(i) $2, 4, 6, 8, \ldots\ldots\ldots\ldots$ Here $a = 2$, $d = 2$, $t_n = 2 + (n-1)2$

(ii) $3, 9, 15, 21, \ldots\ldots\ldots$ Here $a = 3$, $d = 6$, $t_n = 3 + (n-1)6$

1.6.2 Arithmetic Mean (AM): If the quantities a, m, b are in AP then

$$m - a = b - m \Rightarrow m = \frac{a+b}{2}$$ i.e. AM of a and b is $m = \frac{a+b}{2}$

Note: If the quantities $a, m_1, m_2, m_3, \ldots\ldots\ldots, m_n, b$ are in AP then $m_1, m_2, m_3, \ldots\ldots\ldots, m_n$ are called the n arithmetic means of a and b.

1.6.3 Sum of AP series up to n^{th} term : Let a be the 1st term, d be the common difference and l be the last term of an AP series. Let S_n denote the sum of 1st n terms.

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots\ldots\ldots + (l-2d) + (l-d) + l \quad \text{(I)}$$

Also,

$$S_n = l + (l-d) + (l-2d) + \ldots\ldots\ldots + (a+3d) + (a+2d) + (a+d) + a \quad \text{(II)}$$

$$\text{(I)} + \text{(II)} \Rightarrow 2S_n = n(a+l)$$

(iii) Given $\log\frac{x+2y}{4}=\frac{1}{2}(\log x+\log y)$

$$\Rightarrow \log\left(\frac{x+2y}{4}\right)^2=\log x+\log y=\log xy$$

$$\Rightarrow \left(\frac{x+2y}{4}\right)^2=xy \Rightarrow (x+2y)^2=4^2xy=16xy$$

$$\Rightarrow x^2+4y^2=12xy \qquad \text{Proved.}$$

Ex 9. Solve: (i) $\log_x 2+\log_x 4+\log_x 64=9$

(ii) $\log_x(8x-3)-\log_x 4=2$

(iii) **[QP 2014]** $\log(x+1)=2\log x$

Soln: (i) $\log_x 2+\log_x 4+\log_x 64=9$

$$\Rightarrow \log_x(2\times4\times64)=9$$

$$\Rightarrow x^9=2\times4\times64=2^9$$

$$\Rightarrow x=2$$

(ii) $\log_x(8x-3)-\log_x 4=2$

$$\Rightarrow \log_x\left(\frac{8x-3}{4}\right)=2 \quad \Rightarrow x^2=\frac{8x-3}{4}$$

$$\Rightarrow 4x^2-8x+3=0 \quad \Rightarrow (2x-3)(2x-1)=0$$

$$\Rightarrow x=\frac{1}{2},\ \frac{3}{2}$$

(iii) $\log(x+1)=2\log x$

$$\Rightarrow x+1=x^2 \Rightarrow x^2-x-1=0 \quad \Rightarrow x=\frac{1+\sqrt{5}}{2} \qquad \text{since } x>0$$

Ex 10. Prove that $\log_3 8$ is an irrational number.

Soln: Let $x=\log_3 8 \quad \Rightarrow 3^x=8$

If $x=\frac{p}{q}$ where p, q are integers then $3^p=8^q$

This is impossible since 3^p is odd and 8^q is even.

Thus $\log_3 8$ is not a rational number.

Exercise 1.5

1. Find the value of : (i) $\log_6 216$ (ii) $\log_{11} 121$ (iii) $\log_y y$ (iv) $\log_{16} 4$ (v) $\log_{25} 5$

Ans: (i) 3 (ii) 2 (iii) 1 (iv) $\frac{1}{2}$ (v) $\frac{1}{2}$

2. Find the logarithm of: (i) 625 to the base $\sqrt{5}$ (ii) 2025 to the base $3\sqrt{5}$ (iii) 144 to the base $2\sqrt{3}$ (iv) 324 to the base $3\sqrt{2}$

Ans: (i) 8 (ii) 4 (iii) 4 (iv) 4

3. Find the base if the logarithm of: (i) 1728 is 6 (ii) 729 is 4

Ans: (i) $2\sqrt{3}$ (ii) $3\sqrt{3}$

4. Find x if: (i) $\log_5 x=-4$ (ii) $\log_x 81=4$ (iii) $\log_3 x=2$

Ans: (i) $\frac{1}{625}$ (ii) 3 (iii) 9

5. Prove that (i) $\log_2[\log_2(\log_2 256)]=1$ (ii) $\log_6(\log_{\sqrt{2}} 8)=1$

6. Find the value: (i) $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80}$ Ans: log 2

(ii) $\log\frac{25}{33}+\log\frac{55}{46}-\log\frac{65}{69}$ Ans: $\log\frac{25}{26}$

7. Simplify: (i) $\log 2+\log\sqrt{12}+\log\sqrt{\frac{2}{3}}-\log\sqrt{2}$ Ans: 2 log 2

(ii) $\log\frac{a^2b^2}{c^4}+\log\frac{b^2c^2}{a^2}-\log\frac{b^4}{a^2c^3}-\log\frac{a^3b^2}{c^4}+\log\frac{ab^2}{c^5}$ Ans: 0

8. If $a=\log_y z$, $b=\log_z x$, $c=\log_x y$, then prove that $abc=1$

9. Find the value of a if $\log_2 a+\log_4 a+\log_{16} a=\frac{21}{4}$ Ans: $a=8$

$$= \log \frac{2 \times 16^{16} \times 25^{12} \times 81^{7}}{15^{16} \times 24^{12} \times 80^{7}} = \log \frac{2 \times 2^{64} \times 5^{24} \times 3^{28}}{3^{16} \times 5^{16} \times 3^{12} \times 2^{36} \times 2^{28} \times 5^{7}}$$

$$= \log \frac{2^{65} \times 5^{24} \times 3^{28}}{3^{28} \times 5^{23} \times 2^{64}} = \log (2 \times 5)$$

$= \log 10$

$= 1$ (base is not given, so we assume base as 10)

$=$ RHS Proved.

(iii) LHS $= \log \frac{14}{15} + \log \frac{28}{27} + \log \frac{405}{196}$

$$= \log \frac{14 \times 28 \times 405}{15 \times 27 \times 196} = \log \frac{2^3 \times 7^2 \times 3^4 \times 5}{5 \times 3^4 \times 2^2 \times 7^2}$$

$= \log 2 =$ RHS Proved

Ex 7. Simplify: (i) **[QP 2011, 2016]** $\log \frac{81}{8} - 2\log \frac{3}{2} + 3\log \frac{2}{3} + \log \frac{3}{4}$

(ii) **[QP 2014]** $\log \frac{9}{10} + \log \frac{25}{24} - \log \frac{15}{16}$

Soln: (i) $\log \frac{81}{8} - 2\log \frac{3}{2} + 3\log \frac{2}{3} + \log \frac{3}{4}$

$= \log \frac{81}{8} - \log (\frac{3}{2})^2 + \log (\frac{2}{3})^3 + \log \frac{3}{4}$

$= \log 81 + \log 2^2 + \log 2^3 + \log 3 - \log 8 - \log 3^2 - \log 3^3 - \log 4$

$$= \log \frac{81 \times 2^2 \times 2^3 \times 3}{2^3 \times 3^2 \times 3^3 \times 2^2} = \log \frac{3^5 \times 2^5}{3^5 \times 2^5} = \log 1 = 0$$

(ii) $\log \frac{9}{10} + \log \frac{25}{24} - \log \frac{15}{16}$

$= \log 9 + \log 25 + \log 16 - \log 10 - \log 24 - \log 15$

$$= \log \frac{9 \times 25 \times 16}{10 \times 24 \times 15} = \log \frac{3^2 \times 5^2 \times 2^4}{5^2 \times 3^2 \times 2^4} = \log 1 = 0$$

Ex 8. (i) If $a^2 + b^2 = 7ab$ then prove that

a. $\log \frac{a+b}{3} = \frac{1}{2} (\log a + \log b)$

b. $2\log (a-b) = \log 5 + \log a + \log b$

c. $2\log (a+b) = \log 9 + \log a + \log b$

(ii) If $\log \frac{a+b}{3} = \frac{1}{2} (\log a + \log b)$ then prove that $\frac{a}{b} + \frac{b}{a} = 7$

(iii) If $\log \frac{x+2y}{4} = \frac{1}{2} (\log x + \log y)$ then prove that $x^2 + 4y^2 = 12xy$

Soln: (i) a. Given $a^2 + b^2 = 7ab \Rightarrow a^2 + b^2 + 2ab = 9ab$

$\Rightarrow (a+b)^2 = 3^2 (ab)$

$\Rightarrow (\frac{a+b}{3})^2 = ab \Rightarrow \frac{a+b}{3} = \sqrt{ab}$

Taking log on both sides $\log \frac{a+b}{3} = \frac{1}{2} (\log ab) = (\log a + \log b)$ Proved.

b. Given $a^2 + b^2 = 7ab \Rightarrow a^2 + b^2 - 2ab = 5ab$

$\Rightarrow (a-b)^2 = 5ab$

Taking log on both sides $2\log (a-b) = \log (5ab) = \log 5 + \log a + \log b$ Proved.

c. Given $a^2 + b^2 = 7ab \Rightarrow a^2 + b^2 + 2ab = 9ab$

$\Rightarrow (a+b)^2 = 9ab$

Taking log on both sides $2\log (a+b) = \log (9ab) = \log 9 + \log a + \log b$ Proved.

(ii) Given $\log \frac{a+b}{3} = \frac{1}{2} (\log a + \log b)$

$\Rightarrow \log (\frac{a+b}{3})^2 = \log ab \quad (\frac{a+b}{3})^2 = ab$

$\Rightarrow (a+b)^2 = 3^2 (ab) = 9ab \Rightarrow a^2 + b^2 = 7ab$

$\Rightarrow \frac{a}{b} + \frac{b}{a} = 7$ Proved.

(ii) $\log_2 \log_2 16 = \log_2 \log_2 (2)^4 = \log_2 (\log_2 2^4)$

$= \log_2 (4 \log_2 2) = \log_2 (4 \times 1) = \log_2 4 = \log_2 2^2$

$= 2 \log_2 2 = 2 \times 1 = 2$

(iii) $\log_2 \log_3 \log_2 512 = \log_2 \log_3 (\log_2 2^9) = \log_2 \log_3 (9 \log_2 2)$

$= \log_2 \log_3 (9 \times 1) = \log_2 \log_3 9 = \log_2 (\log_3 3^2)$

$= \log_2 (2 \log_3 3) = \log_2 (2 \times 1) = \log_2 2 = 1$

(iv) $\log_2(\sqrt{6}) + \log_2\left(\sqrt{\frac{2}{3}}\right) = \log_2\left(\sqrt{6} \times \sqrt{\frac{2}{3}}\right) = \log_2\left(\sqrt{2} \times \sqrt{3} \times \frac{\sqrt{2}}{\sqrt{3}}\right)$

$= \log_2 2 = 1$

Ex4. Prove that (i) **[QP 2011]** $\log_b a \times \log_c b \times \log_a c = 1$

(ii) **[QP 2009]** $\log_3 \log_2 \log_{\sqrt{3}} 81 = 1$

(iii) **[QP 2011]** $\log_2 \log_{\sqrt{2}} \log_3 81 = 2$

(iv) **[QP 2013]** $\log_2 \log_3 \log_2 512 = 1$

Soln: (i) LHS $= \log_b a \times \log_c b \times \log_a c = (\log_b a \times \log_c b) \times \log_a c$

$= \log_c a \times \log_a c = \log_a a = 1 =$ RHS Proved.

(ii) LHS $= \log_3 \log_2 \log_{\sqrt{3}} 81 = \log_3 \log_2 \log_{\sqrt{3}} (\sqrt{3})^8$

$= \log_3 \log_2 (8 \times 1) = \log_3 \log_2 8 = \log_3 \log_2 2^3$

$= \log_3 (3 \times 1) = \log_3 3 = 1 =$ RHS Proved.

(iii) LHS $= \log_2 \log_{\sqrt{2}} \log_3 81 = \log_2 \log_{\sqrt{2}} \log_3 3^4$

$= \log_2 \log_{\sqrt{2}} (4 \times 1) = \log_2 \log_{\sqrt{2}} (\sqrt{2})^4$

$= \log_2 (4 \times 1) = \log_2 4 = 2 =$ RHS Proved.

(iv) LHS $= \log_2 \log_3 \log_2 512 = \log_2 \log_3 (\log_2 2^9) = \log_2 \log_3 (9 \log_2 2)$

$= \log_2 \log_3 (9 \times 1) = \log_2 \log_3 9 = \log_2 (\log_3 3^2)$

$= \log_2 (2 \log_3 3) = \log_2 (2 \times 1) = \log_2 2 = 1 =$ RHS Proved.

Ex 5. Prove that (i) **[QP 2010]** $\log 1 + \log 2 + \log 3 = \log (1 + 2 + 3)$

(ii) **[QP 2010, 2017]** $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1$

Soln: (i) LHS $= \log 1 + \log 2 + \log 3 = \log (1 \times 2 \times 3)$

$= \log 6 = \log (1 + 2 + 3) =$ RHS Proved.

(ii) $\log (x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y})$

$= \log (x^{\log y - \log z}) + \log (y^{\log z - \log x}) + \log (z^{\log x - \log y})$

$= (\log y - \log z) \log x + (\log z - \log x) \log y + (\log x - \log y) \log z$

$= \log y \times \log x - \log z \times \log x + \log z \times \log y - \log x \times \log y$
$+ \log x \times \log z - \log y \times \log z$

$= 0 = \log 1$

This concludes $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1$

Ex 6. Prove that (i) **[QP 2010, 2013, 2014]** $7 \log \frac{10}{9} - 2 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$

(ii) **[QP 2009, 2016]** $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$

(iii) **[QP 2013]** $\log \frac{14}{15} + \log \frac{28}{27} + \log \frac{405}{196} = \log 2$

Soln: (i) LHS $= 7 \log \frac{10}{9} - 2 \log \frac{25}{24} + 3 \log \frac{81}{80}$

$= \log (10)^7 + \log (24)^2 + \log (81)^3 - \log (9)^7 - \log (25)^2 - \log (80)^3$

$= \log \frac{10^7 \times 24^2 \times 81^3}{9^7 \times 25^2 \times 80^3} = \log \frac{2^7 \times 5^7 \times 3^2 \times 2^6 \times 3^{12}}{3^{14} \times 5^4 \times 5^3 \times 2^{12}}$

$= \log \frac{2^{13} \times 5^7 \times 3^{14}}{3^{14} \times 5^7 \times 2^{12}} = \log 2 =$ RHS Proved.

(ii) LHS $= \log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80}$

$= \log 2 + \log \left(\frac{16}{15}\right)^{16} + \log \left(\frac{25}{24}\right)^{12} + \log \left(\frac{81}{80}\right)^7$

$= \log 2 + \log (16)^{16} + \log (25)^{12} + \log (81)^7 - \log (15)^{16}$
$- \log (24)^{12} - \log (80)^7$

Ex 15. Find the sum: (i) $5+8+11+\ldots\ldots$ to 30^{th} term.

(ii) **[2015]** $2+5+8+11+\ldots\ldots$ to 16^{th} term.

(iii) $-9-1+7+\ldots\ldots$ to 24^{th} term.

(iv) $8+3-2-\ldots\ldots-87$

Soln: (i) Here $a=t_1=5,\ d=t_2-t_1=8-5=3$

Formula for sum is $S_n=\frac{n}{2}\{2a+(n-1)d\}$

$$S_{30}=\frac{30}{2}\{2\times 5+(30-1)\times 3\}=15\times 97=1455$$

(ii) Here $a=t_1=2,\ d=t_2-t_1=5-2=3,\ n=16.$

Formula for sum is $S_n=\frac{n}{2}\{2a+(n-1)d\}$

$$S_{16}=\frac{16}{2}\{2\times 2+(16-1)\times 3\}=392$$

(iii) Here $a=t_1=-9,\ d=t_2-t_1=(-1)-(-9)=8$

Formula for sum is $S_n=\frac{n}{2}\{2a+(n-1)d\}$

$$S_{24}=\frac{24}{2}\{2\times(-9)+(24-1)\times 8\}$$

$$=12\times 166=1992$$

(iv) Here $a=t_1=8,\ d=t_2-t_1=3-8=-5$

We need to find number of terms. Last term is -87

$$t_n=-87=8+(n-1)(-5)\quad\Rightarrow n=20$$

Formula for sum is $S_n=\frac{n}{2}\{2a+(n-1)d\}$

$$S_{20}=\frac{20}{2}\{2\times 8+(20-1)\times(-5)\}$$

$$=10\times(-79)=-790$$

Exercise 1.6(A)

1. Find the series if (i) $t_n=2n^2-2$ (ii) $t_n=\frac{n}{2n+1}$

Ans: (i) 0, 6, 16, 30, ... (ii) $\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{4}{9},\ldots\ldots$

2. The 3^{rd} term of an AP is 18 and 7^{th} term is 30. Find the progression.
Ans: 12, 15, 18, 21, ...

3. The 1^{st} term of an AP is 5 and common difference is 3. The last term is 80. Find the number of terms. Ans: 26

4. (i) The 1^{st} term of an AP is 6 and common difference is 2. Find the 15^{th} term.
(ii) The 6^{th} term and 17^{th} term of an AP are 19 and 41 respectively. Find the 40^{th} term.
Ans: (i) 34 (ii) 87

5. Which term of the AP $4, 5\frac{1}{3}, 6\frac{2}{3}, 8, \ldots\ldots$ is 104? Ans: 76

6. How many terms are there in the AP 7, 13, 19, 25, ..., 205? Ans: 34

7. Fill up the blanks if the following sequences are in AP-
(i) -5, ---, ---, ---, 15. (ii) 34, ---, ---, ---, ---, 48
Ans: (i) 0, 5, 10 (ii) 36.8, 39.6, 42.4, 45.2

8. Insert (i) 4 AMs between 2 and 12.
(ii) 5 AMs between 1.2 and 8.4.
(iii) 3 AMs between -7 and 9.
(iv) 3 AMs between -24 and 0.
(v) 3 AMs between 3 and 19.
(vi) 6 AMs between 15 and -13.

Ans: (i) 4, 6, 8, 10 (ii) 2.4, 3.6, 4.8, 6, 7.2 (iii) -3, 1, 5
(iv) $-18, -12, -6$ (v) 7, 11, 15 (vi) 11, 7, 3, -1, -5, -9

$t_p = a = \frac{p}{2}\{2l + (p-1)m\},\ t_q = b = \frac{q}{2}\{2l + (q-1)m\},\ t_r = c = \frac{r}{2}\{2l + (r-1)m\}.$

$\Rightarrow \frac{a}{p}(q-r) = l(q-r) + \frac{1}{2}(p-1)(q-r)m$

$\Rightarrow \frac{b}{q}(r-p) = l(r-p) + \frac{1}{2}(q-1)(r-p)m$

$\Rightarrow \frac{c}{r}(p-q) = l(p-q) + \frac{1}{2}(r-1)(p-q)m$ Adding all the three we have

$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0$

Ex 10. If in an AP p^{th} term is q and q^{th} term p then prove that n^{th} term is $(p+q-n)$.

Soln: Let a be the 1st term and d be the common difference.

Given $t_p = q = a + (p-1)d$ and $t_q = p = a + (q-1)d$

From this we have $q - p = (p-q)d$ or $d = -1$

And $a = p - (q-1)d = p - (q-1)(-1) = p + q - 1$

After having a and d we have $t_n = (p+q-1) + (n-1)(-1)$

$= p + q - n$

Ex 11. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in AP, then prove that a^2, b^2, c^2 are in AP.

Soln: $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in AP $\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$\Rightarrow \frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)}$

$\Rightarrow \frac{(b-a)(a+b)}{(c+a)(b+c)(a+b)} = \frac{(c-b)(b+c)}{(a+b)(c+a)(b+c)}$

$\Rightarrow \frac{b^2-a^2}{(c+a)(b+c)(a+b)} = \frac{c^2-b^2}{(a+b)(c+a)(b+c)}$

$\Rightarrow b^2 - a^2 = c^2 - b^2$

$\Rightarrow a^2, b^2, c^2$ are in AP.

Ex 12. If $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP then prove that a, b, c, are in AP or $ab + bc + ca = 0$.

Soln: $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP

$\Rightarrow b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

$\Rightarrow b^2c + b^2a - a^2b - a^2c = c^2a + c^2b - b^2c - b^2a$

$\Rightarrow c(b^2 - a^2) + ab(b-a) = a(c^2 - b^2) + bc(c-b)$

$\Rightarrow (b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$

$\Rightarrow$ either $b - a = c - b$ when $ab + bc + ca \neq 0$

or $ab + bc + ca = 0$

Ex 13. Determine the value of k if following are in AP.

(i)[QP 2010] $7k+3, 4k-5, 2k+10$ **(ii)[QP 2016]** $2k+3, 3k+1, 5k+3$

Soln: $7k+3, 4k-5, 2k+10$ are in AP

$\Rightarrow (4k-5) - (7k+3) = (2k+10) - (4k-5)$

$\Rightarrow -3k - 8 = -2k + 15$

$\Rightarrow k = -23$ Numbers are $-158, -97, -36$

(ii) $2k+3, 3k+1$ and $5k+3$ are in AP

$\Rightarrow (3k+1) - (2k+3) = (5k+3) - (3k+1)$

$\Rightarrow k - 2 = 2k + 2 \Rightarrow k = -4$. Numbers are $-5, -11, -17$.

Ex 14. Find the AP, the sum of whose n terms is (i) $n^2 + 3n$ (ii) [QP 2013] $2n^2 + 5n$

Soln: (i) $S_n = n^2 + 3n$ so $S_1 = t_1 = 4,\ S_2 = t_1 + t_2 = 10 \Rightarrow t_2 = 6$

$S_3 = t_1 + t_2 + t_3 = 18 \Rightarrow t_3 = 8,$

$S_4 = t_1 + t_2 + t_3 + t_4 = 28 \Rightarrow t_4 = 10$

Generally $t_n = S_n - S_{n-1} = (n^2 + 3n) - \{(n-1)^2 + 3(n-1)\} = 2n + 2$

The AP is 4, 6, 8, 10,......................

(ii) $S_n = 2n^2 + 5n$ so $S_1 = t_1 = 7,\ S_2 = t_1 + t_2 = 18 \Rightarrow t_2 = 11$

$S_3 = t_1 + t_2 + t_3 = 33 \Rightarrow t_3 = 15,$

$S_4 = t_1 + t_2 + t_3 + t_4 = 52 \Rightarrow t_4 = 19$

Generally $t_n = S_n - S_{n-1} = (2n^2 + 5n) - \{2(n-1)^2 + 5(n-1)\} = 4n + 3$

The AP is 7, 11, 15, 19, ...

$1+(-\frac{15}{64})=\frac{49}{64}, 1+2.-\frac{15}{64}=\frac{34}{64}=\frac{17}{32}$, and $1+3.-\frac{15}{64}=\frac{19}{64}$.

(v) $6, m_1, m_2, m_3, 18$ are in AP.

$a=6,\quad t_5=18=6+(5-1)\,d$

$\Rightarrow d=3$. Hence the 3 AMs are 9, 12, 15.

(vi) $4, m_1, m_2, m_3, m_4, 19$ are in AP.

$a=4,\quad t_6=19=4+(6-1)\,d$

$\Rightarrow d=3$. Hence the 4 AMs are 7, 10, 13 and 16.

(vii) $3, m_1, m_2, m_3, m_4, 23$ are in AP.

$a=3,\quad t_6=23=3+(6-1)\,d$

$\Rightarrow d=4$. Hence the 4 AMs are 7, 11, 15 and 19.

(viii) $3, m_1, m_2, m_3, 19$ are in AP.

$a=3,\quad t_5=19=3+(5-1)\,d$

$\Rightarrow d=4$. Hence the 3 AMs are 7, 11, 15.

Ex 5. Find the AP if (i) 5^{th} term is 23 and 11^{th} term is 47.

(ii) 10^{th} term is 41 and 18^{th} term is 73.

Soln: (i) Let a be the 1^{st} term and d be the common difference.

Given $t_5=23=a+(5-1)\,d$ and $t_{11}=47=a+(11-1)\,d$

We have $23=a+4d$ (I) and $47=a+10d$ (II)

(II) – (I) $\Rightarrow 6d=24 \Rightarrow d=4$ from (I) $a=23-4\times 4=7$

. Now the AP is 7, 11, 15, 19,............

(ii) Let a be the 1^{st} term and d be the common difference.

Given $t_{10}=41=a+(10-1)\,d$ and $t_{18}=73=a+(18-1)\,d$

We have $41=a+9d$ (I) and $73=a+17d$ (II)

(II) – (I) $\Rightarrow 8d=32 \Rightarrow d=4$ from (I) $a=41-9\times 4=5$

. Now the AP is 5, 9, 13, 17,............

Ex 6. [QP 2010, 2016] If 12^{th} term and 15^{th} term of an AP are 68 and 86 respectively, find its 18^{th} term.

Soln: Let a be the 1^{st} term and d be the common difference.

Given $t_{12}=68=a+(12-1)\,d$ and $t_{15}=86=a+(15-1)\,d$

We have $68=a+11d$ (I) and $86=a+14d$ (II)

(II) – (I) $\Rightarrow 3d=18 \Rightarrow d=6$ from (I) $a=68-11\times 6=2$

Now 18^{th} term is $t_{18}=2+(18-1)\times 6=104$

Ex 7. The 9^{th} term of an AP is 0. Prove that 29^{th} term is double the 19^{th} term.

Soln: Let a be the 1^{st} term and d be the common difference.

Given $t_9=0=a+(9-1)\,d \Rightarrow a=-8d$

Now $t_{19}=a+18d=10d+(a+8d)=10d+0=10d$

and $t_{29}=a+28d=20d+(a+8d)=20d+0=20d=2\times t_{19}$ proved.

Ex 8. In an AP $t_3 : t_5 = 1:4$; show that $t_7 : t_{12} = 14:29$.

Soln: Let a be the 1^{st} term and d be the common difference.

We have $t_3=a+2d,\ t_5=a+4d,\ t_7=a+6d,\ t_{12}=a+11d$

Given $\frac{a+2d}{a+4d}=\frac{1}{4} \Rightarrow 4a+8d=a+4d \Rightarrow a=-\frac{4}{3}d$

We see $t_7=-\frac{4}{3}d+6d=\frac{14}{3}d$ (I)

and $t_{12}=-\frac{4}{3}d+11d=\frac{29}{3}d$ (II)

Now from (I) and (II) $t_7 : t_{12}=\frac{14}{3}d : \frac{29}{3}d=14:29$ proved.

Ex 9 (i). [QP 2013] If a, b, c, are the p^{th}, q^{th} and r^{th} terms of an AP, prove that $a(q-r)+b(r-p)+c(p-q)=0$

(ii). [QP 2015] The sum of p, q, r terms of an AP are a, b, c respectively. Show that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$

Soln (i): Let m be the 1^{st} term and d be the common difference. Given

$t_p=a=m+(p-1)\,d \Rightarrow a(q-r)=mq-mr+d(pq-rp-q+r)$ (I)

$t_q=b=m+(q-1)\,d \Rightarrow b(r-p)=mr-mp+d(qr-pq-r+p)$ (II)

$t_r=c=m+(r-1)\,d \Rightarrow c(p-q)=mp-mq+d(rp-qr-p+q)$ (III)

It is clear that (I) + (II) + (III) $\Rightarrow a(q-r)+b(r-p)+c(p-q)=0$

(ii) Let first term be l and common difference be m. We have

$$\Rightarrow S_n = \frac{n}{2}(a+l) = \frac{n}{2}[a+\{a+(n-1)d\}] = \frac{n}{2}\{2a+(n-1)d\}$$

Thus $S_n = \frac{n}{2}\{2a+(n-1)d\}$

Worked out examples:

Ex 1. (i) Find 8^{th} term of $\frac{5}{3}, \frac{7}{3}, 3, 3\frac{2}{3}, \ldots\ldots\ldots$

(ii) **[QP 2012]** Find 11^{th} term of .09, .17, .25, .33,.........

(iii) Find 10^{th} term of 40, 30, 20, 10, ...

Soln: (i) $a = \frac{5}{3}$, $d = \frac{7}{3} - \frac{5}{3} = \frac{2}{3}$, $t_8 = \frac{5}{3} + (8-1)\frac{2}{3} = \frac{19}{3}$

(ii) $a = .09$, $d = .17 - .09 = .08$, $t_{11} = .09 + (11-1)\times .08 = .89$

(iii) $a = 40$, $d = 30 - 40 = -10$, $t_{10} = 40 + (10-1)(-10) = -50$

Ex 2. [QP 2009] Is 292 a term of AP series 1, 4, 7, 10,.................. ?

Soln: $a = 1$, $d = 4 - 1 = 3$,

We assume $t_n = 292 = 1 + (n-1)\times 3 \Rightarrow 3n - 2 = 292$

$\Rightarrow n = 98$

It is concluded that 292 is the 98^{th} term of the given AP.

Ex 3. (i) How many numbers divisible by 17 are there between 25 and 450?

(ii) Find the number of integers between 500 and 2000 that are not divisible by 5.

Soln: (i) The number divisible by 17 and greater than 25 is 34. And the number divisible by 17 and less than 450 is 442. The terms in the AP

34, 51, 68, ...442 are all the numbers divisible by 17.

We need to find number of terms.

Here $a = 34$, $d = 17$, $t_n = 442 = 34 + (n-1)\times 17$

$\Rightarrow n = (408/17) + 1 = 25$

So there are 25 numbers divisible by 17 in between 25 and 450.

(ii) The terms in the AP: 500, 505, 510,........., 2000 are the numbers divisible by 5. Here $a = 500$, $d = 5$, $t_n = 2000 = 500 + (n-1)\times 5$

$\Rightarrow n = (1500/5) + 1 = 301$

So there are 301 numbers divisible by 5 lying in [500, 2000].

There is a total of 1501 numbers. Hence number of integers not divisible by 5 is

$1501 - 301 = 1200$

Ex 4. Insert (i) 10 AMs between 2 and 57.

(ii) 3 AMs between 5 and 17.

(iii) **[QP 2009, 2010]** 5 AMs between 2 and 23.

(iv) **[QP 2009, 2016]** 3 AMs between 1 and $\frac{1}{16}$.

(v) **[QP 2010]** 3 AMs between 6 and 18.

(vi) **[QP 2011]** 4 AMs between 4 and 19.

(vii) **[QP 2011]** 4 AMs between 3 and 23.

(viii) **[QP 2013]** 3 AMs between 3 and 19.

Soln: (i) $2, m_1, m_2, \ldots\ldots\ldots\ldots m_{10}, 57$ are in AP.

$a = 2$, $t_{12} = 57 = 2 + (12-1)d$

$\Rightarrow d = 5$. Hence the 10 AMs are 7, 12, 17, 22, 27, 32, 37, 42, 47 and 52.

(ii) $5, m_1, m_2, m_3, 17$ are in AP.

$a = 5$, $t_5 = 17 = 5 + (5-1)d$

$\Rightarrow d = 3$. Hence the 3 AMs are 8, 11, 14.

(iii) $2, m_1, m_2, m_3, m_4, m_5, 23$ are in AP.

$a = 2$, $t_7 = 23 = 2 + (7-1)d$

$\Rightarrow d = 3\frac{1}{2}$. Hence the 5 AMs are $5\frac{1}{2}, 9, 12\frac{1}{2}, 16$ and $19\frac{1}{2}$.

(iv) $1, m_1, m_2, m_3, \frac{1}{16}$ are in AP.

$a = 1$, $t_5 = \frac{1}{16} = 1 + (5-1)d$

$\Rightarrow d = (\frac{1}{16} - 1)/4 = -\frac{15}{64}$. Hence the 3 AMs are from (II) / (I) we get $r^4 = \frac{1}{16} = \left(\frac{1}{2}\right)^4 \Rightarrow r = \frac{1}{2}$

from (I) $a = 1/(r^6) = 2^6 = 64$

Now the GP is 64, 32, 16, 8,..........

Ex 6. Find the three terms of the GP whose sum is

(i) 52 and whose product in pairs is 624.

(ii) **[QP 2014]** 21 and product is 64.

(iii) **[QP 2014]** 19 and product is 216.

(iv) **[QP 2013]** 49 and product is 512.

(v) **[QP 2015]** 35 and their product is 1000

Soln: Let the terms be a, ar, ar^2

(i) Given $a + ar + ar^2 = 52$,

$\Rightarrow a(1 + r + r^2) = 52$ (I)

$a \times ar + ar \times ar^2 + ar^2 \times a = 624$

$\Rightarrow a^2 r(1 + r^2 + r) = 624$ (II)

from (II) / (I) we get $ar = 12$

In (I) we write $a + 12 + (144/a) = 52$ which yields $a^2 - 40a + 144 = 0$

$\Rightarrow (a-4)(a-36) = 0 \Rightarrow a = 4, r = 3; a = 36, r = 1/3$

So GP is 4, 12, 36 or 36, 12, 4

(ii) Given $a + ar + ar^2 = 21$ (I)

$a \times ar \times ar^2 = 64$

$\Rightarrow a^3 r^3 = 64 \Rightarrow ar = 4$

In (I) we write $a + 4 + (16/a) = 21$ which yields $a^2 - 17a + 16 = 0$

$\Rightarrow (a-1)(a-16) = 0 \Rightarrow a = 1, r = 4; a = 16, r = 1/4$

So GP is 1, 4, 16 or 16, 4, 1

(iii) Given $a + ar + ar^2 = 19$ (I)

$a \times ar \times ar^2 = 216$

$\Rightarrow a^3 r^3 = 216 \Rightarrow ar = 6$

In (I) we write $a + 6 + (36/a) = 19$ which yields $a^2 - 13a + 36 = 0$

$\Rightarrow (a-9)(a-4) = 0 \Rightarrow a = 9, r = 2/3; a = 4, r = 3/2$

So GP is 9, 6, 4 or 4, 6, 9

(iv) Given $a + ar + ar^2 = 49$ (I)

$a \times ar \times ar^2 = 512$

$\Rightarrow a^3 r^3 = 512 \Rightarrow ar = 8$

In (I) we write $a + 8 + (64/a) = 49$ which yields $a^2 - 41a + 64 = 0$

$$\Rightarrow a = \frac{41 \pm \sqrt{1681 - 256}}{2} = \frac{41 \pm \sqrt{1425}}{2}$$

Thus $a = \frac{41 + 5\sqrt{57}}{2}$, $r = \frac{16}{41 + 5\sqrt{57}}$; $a = \frac{41 - 5\sqrt{57}}{2}$, $r = \frac{16}{41 - 5\sqrt{57}}$.

So GP is $\frac{41 + 5\sqrt{57}}{2}, 8, \frac{128}{41 + 5\sqrt{57}}$ or $\frac{41 - 5\sqrt{57}}{2}, 8, \frac{128}{41 - 5\sqrt{57}}$

(v) The three numbers are a, ar, ar^2. Given $a^3 r^3 = 1000$

$\Rightarrow ar = 10$ Also given $a + ar + ar^2 = 35$

$\Rightarrow a + 10 + \frac{100}{a} = 35$

$\Rightarrow a^2 - 25a + 100 = 0 \Rightarrow a = 5, 20$. Hence the numbers are 5, 10, 15.

Ex 7. Insert (i) 3 GMs between 9 and $\frac{1}{9}$.

(ii) 5 GMs between 576 and 9.

(iii) **[QP 2015]** 4 GM between 4 and $\frac{1}{256}$

Soln: (i) Let GMs are M_1, M_2, M_3. So $9, M_1, M_2, M_3, \frac{1}{9}$ are in GP.

$a = 9$, $t_5 = \frac{1}{9} = ar^4 = 9r^4 \Rightarrow r^4 = \frac{1}{81}$. $\Rightarrow r = \frac{1}{3}$

Hence the 3 GMs are $3, 1, \frac{1}{3}$

(ii) Let GMs are M_1, M_2, M_3, M_4, M_5.

So $576, M_1, M_2, M_3, M_4, M_5, 9$ are in GP.

Worked out examples:

Ex 1. Find (i) 12^{th} term of $-6, 18, -54, \ldots\ldots$

(ii) 10^{th} term of $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\ldots$

(iii) 8^{th} term of $4, -288, 20736, \ldots\ldots$

Soln: (i) $t_1 = a = -6, \quad r = -3, \quad t_{12} = (-6)(-3)^{11} = 6\times 3^{11}$

(ii) $t_1 = a = 3, \quad r = \frac{1}{2}, \quad t_{10} = 3\times\left(\frac{1}{2}\right)^9$

(iii) $t_1 = a = 4, \quad r = -72, \quad t_8 = 4\times(-72)^7 = -4\times(72)^7$

Ex 2. What term of the series $2, 4, 8, 16, \ldots\ldots$ is 2048?

Soln: $t_1 = a = 2, \quad r = 2, \quad t_n = 2\times 2^{n-1} = 2^n = 2048 = 2^{11}$

Hence 2048 is the 11^{th} term.

Ex 3. (i) **[QP 2013]** Examine if 222 is a term of the series $1, 3, 9, \ldots\ldots$

(ii) Is 502 a term of the progression $2, 8, 32, \ldots\ldots$? If not find the term of the progression nearest to it.

Soln: (i) $t_1 = a = 1, \quad r = 3, \quad$ Let $t_n = 1\times 3^{n-1} = 3^{n-1} = 222 = 2.3.37$

Thus 222 is not a power of 3.

$\Rightarrow$ 222 is not a term of the series.

(ii) $t_1 = a = 2, \quad r = 4, \quad$ Let $t_n = 2\times 4^{n-1} = 2^{2n-1} = 502 = 2.251$

Thus 502 is not a power of 2.

$\Rightarrow$ 502 is not a term of the series.

The number nearest to 502 which is an odd power of 2 is

$$2^9 = 512 = 2\times 4^{5-1} = t_5$$

Ex 4. For what value of a the following may be in GP:

(i) $3a+1, 6a-4, 3a-2$ (ii) **[QP 2009, 2016]** $2a+1, a, 3a+2$

Soln: (i) If $3a+1, 6a-4, 3a-2$ are in GP then

$$\frac{6a-4}{3a+1} = \frac{3a-2}{6a-4}$$

$\Rightarrow (6a-4)^2 = (3a+1)(3a-2)$

$\Rightarrow 36a^2 - 48a + 16 = 9a^2 - 3a - 2$

$\Rightarrow 27a^2 - 45a + 18 = 0$

$\Rightarrow 27a^2 - 27a - 18a + 18 = 0$

$\Rightarrow 27a(a-1) - 18(a-1) = 0$

$\Rightarrow (3a-2)(a-1) = 0 \quad a = 1$ or $a = \frac{2}{3}$ Answer is $a = 1, a = \frac{2}{3}$

For $a = 1$ the GP is 4, 2, 1.

(ii) If $2a+1, a, 3a+2$ are in GP then

$$\frac{a}{2a+1} = \frac{3a+2}{a}$$

$\Rightarrow a^2 = (2a+1)(3a+2)$

$\Rightarrow a^2 = 6a^2 + 7a + 2$

$\Rightarrow 5a^2 + 7a + 2 = 0$

$\Rightarrow 5a^2 + 5a + 2a + 2 = 0$

$\Rightarrow 5a(a+1) + 2(a+1) = 0$

$\Rightarrow (5a+2)(a+1) = 0 \quad a = -1$ or $a = -1$ or $a = -\frac{2}{5}$

For $a = -1$ the GP is $-1, -1, -1$. For $a = -\frac{2}{5}$ GP is $\frac{1}{5}, -\frac{2}{5}, \frac{4}{5}$

Ex 5. (i) The 4^{th} term of a GP is 27 and 7^{th} term is 729. Find the GP.

(ii) The 7^{th} term of a GP is 1 and 11^{th} term is $\frac{1}{16}$. Find the GP.

Soln: (i) $t_1 = a, \quad$ common ratio $= r$.

$t_4 = ar^3 = 27 \quad$ (I), $\quad t_7 = ar^6 = 729 \quad$ (II)

from (II) / (I) we get $r^3 = 729/27 = 27 = 3^3 \Rightarrow r = 3$

from (I) $a = 27/(r^3) = 27/27 = 1$

Now the GP is $1, 3, 9, 27, \ldots\ldots$

(ii) $t_1 = a, \quad$ common ratio $= r$.

$t_7 = ar^6 = 1 \quad$ (I), $\quad t_{11} = ar^{10} = \frac{1}{16} \quad$ (II)

$$\text{Now } A-G=\frac{a+b}{2}-\sqrt{ab}=\frac{a+b-2\sqrt{ab}}{2}=\frac{(\sqrt{a})^2+(\sqrt{b})^2-2\sqrt{a}\sqrt{b}}{2}$$

$$=\frac{(\sqrt{a}-\sqrt{b})^2}{2}>0 \quad \Rightarrow A-G>0 \qquad \Rightarrow A>G$$

1.6.8 Sum of GP series up to n^{th} term: Let a be the 1^{st} term, r be the common ratio

Then $S_n=a+ar+ar^2+ar^3+\ldots\ldots\ldots\ldots+ar^{n-1}$ (I)

Multiplying both sides by r we have

$rS_n=ar+ar^2+ar^3+\ldots\ldots\ldots\ldots+ar^{n-1}+ar^n$ (II)

$(II)-(I) \Rightarrow rS_n-S_n=a(r^n-1)$

$$\Rightarrow S_n=a\,\frac{r^n-1}{r-1}$$

Note: Sum of a GP series up to infinity exists if $|r|<1$ or $-1<r<1$

$S=a+ar+ar^2+ar^3+\ldots\ldots\ldots\ldots+ar^{n-1}+\ldots\ldots\ldots\ldots\text{to }\infty$

$$=\frac{a}{1-r}$$

1.6.9 Natural Numbers :

Let us find the following sum

A. Sum of first n natural numbers.

B. Sum of squares of first n natural numbers.

C. Sum of cubes of first n natural numbers.

A. $1+2+3+\ldots\ldots\ldots\ldots\ldots+n$

$=\frac{n}{2}\{2+(n-1)\times 1\}$ since it is an AP with $a=1$ and $d=1$

$$=\frac{n(n+1)}{2}$$

Thus $\sum_{r=1}^{n} r=\frac{n(n+1)}{2}$

B. To find this sum we consider the identity

$$(k+1)^3-k^3=\{(k+1)-k\}\{(k+1)^2+(k+1)k+k^2\}$$
$$=3k^2+3k+1$$

We write
$$2^3-1^3=3\times1^2+3\times1+1$$
$$3^3-2^3=3\times2^2+3\times2+1$$
$$4^3-3^3=3\times3^2+3\times3+1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n+1)^3-n^3=3n^2+3n+1$$

Adding vertically we get

$$(n+1)^3-1^3=3(1^2+2^2+3^2+\ldots\ldots\ldots\ldots n^2)+3(1+2+3+\ldots\ldots\ldots\ldots+n)+1\times n$$

$$=3(1^2+2^2+3^2+\ldots\ldots\ldots\ldots n^2)+\frac{3n(n+1)}{2}+n$$

$$\Rightarrow \sum_{r=1}^{n} r^2=1^2+2^2+3^2+\ldots\ldots\ldots\ldots n^2$$

$$=\frac{1}{3}\{(n+1)^3-1^3-\frac{3n(n+1)}{2}-n\}=\frac{n(n+1)(2n+1)}{6}$$

C. We know $(k+1)^2-(k-1)^2=4k$

Hence $(k+1)^2k^2-k^2(k-1)^2=4k^3$

Now we can write
$$2^2\times1^2-1^2\times0^2=4\times1^3$$
$$3^2\times2^2-2^2\times1^2=4\times2^3$$
$$4^2\times3^2-3^2\times2^2=4\times3^3$$
$$5^2\times4^2-4^2\times3^2=4\times4^3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n+1)^2n^2-n^2(n-1)^2=4n^3$$

Adding vertically we get

$$(n+1)^2n^2-1^2\times0^2=4(1^3+2^3+3^3+\ldots\ldots\ldots\ldots n^3)$$

$$\Rightarrow \sum_{r=1}^{n} r^3=1^3+2^3+3^3+\ldots\ldots\ldots\ldots n^3=\left(\frac{n(n+1)}{2}\right)^2$$

9. If in an AP r^{th} term is n and n^{th} term is r then prove that m^{th} term is $n-m+r$.
10. If in an AP m times of m^{th} term and n times of n^{th} term are equal then prove that its $(m+n)^{th}$ term is zero.
11. If a, b, c are in AP then prove that
(i) $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in AP.
(ii) $b+c, c+a, a+b$ are also in AP.
(iii) $\{(b+c)^2-a^2\}, \{(c+a)^2-b^2\}, \{(a+b)^2-c^2\}$ are also in AP.
12. If a^2, b^2, c^2 are in AP then prove that
(i) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in AP.
(ii) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in AP.
13. Find the sum-
(i) $3+7+11+\ldots\ldots\ldots\ldots+79$
(ii) $4+8+12+\ldots\ldots\ldots\ldots+80$
(iii) $1+4+7+\ldots\ldots\ldots\ldots+37$
(iv) $12+15+18+\ldots\ldots\ldots\ldots$ to 25 terms.
(v) $0.01+0.04+0.07+\ldots\ldots\ldots$ to 40 terms.
(vi) $3+13+23+\ldots\ldots\ldots\ldots$ to 50 terms.

Ans: (i) 820 (ii) 840 (iii) 247 (iv) 1200 (v) 23.8 (vi) 12400

14. Sum of how many terms of the AP $27+24+21+\ldots\ldots\ldots$ is 132? Give reason for two answers.

Ans: $n=8, n=11$. reason- sum of the last 3 terms is zero.

• • • •

1.6.4 Geometric Progression : A Geometric Progression (GP) is a series of quantities when the ratio of any term (except the first) to the preceding one is constant. This constant is called the common ratio (c.r.) and it is usually denoted by r. As in case of AP we call the first term a and the n^{th} term t_n.

The General form of a GP is $a, ar, ar^2, ar^3, \ldots\ldots\ldots$

Here $t_1 = a$, common ratio $= r$, $t_n = ar^{n-1}$

It is to be noted that $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots\ldots\ldots = \frac{t_{n+1}}{t_n} = r =$ **common ratio**

Note: 1. If any term (except the first) is divided by the immediate preceding term then the common ratio can be determined.

2. If the first term is multiplied by the common ratio then the second term can be obtained; if the second term is multiplied by the common ratio then the third term can be obtained; and so on.

Examples of GP series

(i) $3, 6, 12, 24, \ldots\ldots\ldots$ Here $a=3, r=2, t_n = 3\times 2^{n-1}$

(ii) $1, 4, 16, 64, \ldots\ldots\ldots$ Here $a=1, r=4, t_n = 1\times 4^{n-1}$

1.6.5 Properties of GP: If the terms of a GP are multiplied or divided by a constant then the products are also in GP.

1.6.6 Geometric Mean (GM): Let the three quantities a, G, b are in GP. Then G is said to be Geometric Mean of a and b. So we have $a : G = G : b \Rightarrow G^2 = ab$.

$$\Rightarrow G = \sqrt{ab}$$

Note: 1. The geometric mean between the two quantities is the *square root* of their product.

2. a, b are two given numbers. If $G_1, G_2, G_3, \ldots\ldots, G_n$ are inserted between a and b such that $a, G_1, G_2, G_3, \ldots\ldots, G_n, b$ are in GP then the n numbers $G_1, G_2, G_3, \ldots\ldots, G_n$ are known as n geometric means between a and b.

1.6.7 To prove that AM > GM

Let us consider two numbers a and b.

Let A and G be the AM and GM between the two numbers a and b.

Then $A = \frac{a+b}{2}$, $G = \sqrt{ab}$.

Now $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2}$

$$\Rightarrow r^{10} + r^7 t + r^4 t^2 + \dots\dots\dots\dots + \frac{t^9}{r^{17}}$$

$$= S_{10} = r^{10} \times \frac{(t/r^3)^{10} - 1}{(t/r^3) - 1} = r^{13} \times \frac{t^{10} - r^{30}}{(t - r^3) r^{30}}$$

(xi) $4 + 44 + 444 + \dots\dots\dots$

$$= \frac{4}{9}(9 + 99 + 999 + \dots\dots)$$

$$= \frac{4}{9}\{(10 - 1) + (10^2 - 1) + (10^3 - 1) + \dots\dots\}$$

$$= \frac{4}{9}\{(10 + 10^2 + 10^3 + \dots\dots + 10^n) - (1 + 1 + \dots\dots n \text{ times})\}$$

$$= \frac{4}{9}\{10\,\frac{10^n - 1}{10 - 1} - n\}$$

(xii) $7 + 77 + 777 + \dots\dots\dots$

$$= \frac{7}{9}(9 + 99 + 999 + \dots\dots\dots)$$

$$= \frac{7}{9}\{(10 - 1) + (100 - 1) + (1000 - 1) + \dots\dots\dots + (10^n - 1)\}$$

$$= \frac{7}{9}\{10 + 10^2 + 10^3 + \dots\dots + 10^n - n\}$$

$$= \frac{7}{9}\{10\frac{10^n - 1}{10 - 1} - n\}$$

(xiii) $.9 + .99 + .999 + \dots\dots\dots$

$$= (1 - .1) + (1 - .01) + (1 - .001) + \dots\dots\dots$$

$$= (1 + 1 + \dots\dots n \text{ times}) - \{(.1) + (.1)^2 + (.1)^3 + \dots\dots\dots\}$$

$$= n - (.1) \times \frac{(.1)^n - 1}{(.1) - 1}$$

(xiv) $1.2.5 + 2.3.6 + 3.4.7 + \dots\dots\dots$

$$= 1.2.5 + 2.3.6 + 3.4.7 + \dots\dots + r(r + 1)(r + 4) + \dots\dots\dots$$

$$= \sum_{r=1}^{n} r(r + 1)(r + 4)$$

$$= \sum_{r=1}^{n}(r^3 + 5r^2 + 4r) = \left(\frac{n(n+1)}{2}\right)^2 + 5\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2}$$

(xv) $1.4 + 2.9 + 3.16 + \dots\dots\dots$

$$= 1.4 + 2.9 + 3.16 + \dots\dots\dots + r(r + 1)^2 + \dots\dots\dots$$

$$= \sum_{r=1}^{n} r.(r + 1)^2 = \sum_{r=1}^{n}(r^3 + 2r^2 + r)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

(xvi) $3 + 5 + 9 + 15 + 23 + \dots\dots\dots$

$$= 3 + (3 + 2) + (3 + 2 + 2.2) + (3 + 2 + 2.2 + 2.3) + (3 + 2 + 2.2 + 2.3 + 2.4) + \dots\dots\dots$$

$$= 3 + \{3 + 2.1\} + \{3 + 2(1 + 2)\} + \{3 + 2(1 + 2 + 3)\} + \{3 + 2(1 + 2 + 3 + 4)\} + \dots\dots\dots$$

$$= 3 + (3 + 2.1) + (3 + 3.2) + (3 + 4.3) + (3 + 5.4) + \dots\dots\dots + \{3 + r(r - 1)\} + \dots\dots\dots$$

$$= \sum_{r=1}^{n}\{3 + r(r - 1)\} = 3n + \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$$

$$= 3n + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

Ex 15. If $y = 1 + x + x^2 + \dots\dots\dots$ to infinity. prove that $x = \frac{y-1}{y}$

Soln: We assume $|x| < 1$. So y is the sum of an infinite GP and $y = \frac{1}{1-x}$

$$\Rightarrow y - xy = 1 \Rightarrow xy = y - 1 \Rightarrow x = \frac{y-1}{y}$$

Soln: (i) $t_1 = a = 1$, c.r. $= r = 3$, $S_n = a\dfrac{r^n - 1}{r - 1}$

$$S_{11} = 1 \times \frac{3^{11} - 1}{3 - 1} = \frac{1}{2}(3^{11} - 1)$$

(ii) $t_1 = a = 2$, c.r. $= r = 2$, $S_n = a\dfrac{r^n - 1}{r - 1}$

$$S_8 = 2 \times \frac{2^8 - 1}{2 - 1} = 2\,(2^8 - 1) = 510$$

(iii) $t_1 = a = 3$, c.r. $= r = \dfrac{1}{9}$, $S_n = a\dfrac{r^n - 1}{r - 1}$

$$S_6 = 3 \times \frac{(1/9)^6 - 1}{(1/9) - 1} = 3 \times \frac{9}{8} \times \frac{9^6 - 1}{9^6} = \frac{3}{8 \times 9^5}(9^6 - 1)$$

iv) $t_1 = a = 2$, c.r. $= r = -3$, $S_n = a\dfrac{r^n - 1}{r - 1}$

$$S_{15} = 2 \times \frac{(-3)^{15} - 1}{(-3) - 1} = \frac{1}{2}(3^{15} + 1)$$

(v) $t_1 = a = 2$, c.r. $= r = -\dfrac{3}{8}$, $S_n = a\dfrac{r^n - 1}{r - 1}$

$$S_{10} = 2 \times \frac{\left(-\frac{3}{8}\right)^{10} - 1}{-3/8 - 1} = 2 \times \frac{8}{1} \times \frac{8^{10} - 3^{10}}{8^{10}} = \frac{2}{11} \times \frac{8^{10} - 3^{10}}{8^9}$$

(vi) $t_1 = a = 1$, c.r. $= r = \dfrac{1}{4}$, $S_\infty = S = \dfrac{a}{1 - r}$

$$S = \frac{a}{1 - r} = \frac{1}{1 - (1/4)} = 1 \times \frac{4}{3} = \frac{4}{3}$$

(vii) $t_1 = a = 9$, c.r. $= r = -\dfrac{4}{9}$, $S_\infty = S = \dfrac{a}{1 - r}$

$$S = \frac{9}{1 + 4/9} = 9 \times \frac{9}{13} = \frac{81}{13}$$

(viii) $t_1 = a = 6$, c.r. $= r = \dfrac{1}{2}$,

$$t_n = ar^{n-1} = 6 \times \left(\frac{1}{2}\right)^{n-1} = .09375 = 6 \times .015625$$

$$= 6 \times \frac{15625}{10^6} = 6 \times \frac{5^6}{10^6} = 6 \times \left(\frac{1}{2}\right)^6 = t_7$$

$\Rightarrow t_7 = .09375 \Rightarrow 6 + 3 + 1.5 + .75 + \ldots\ldots\ldots + .09375$

$$= S_7 = a\,\frac{r^n - 1}{r - 1} = 6 \times \frac{(1/2)^7 - 1}{(1/2) - 1} = 6 \times 2 \times \frac{2^7 - 1}{2^7} = 3 \times \frac{2^7 - 1}{2^5}$$

(ix) $t_1 = a = 10$, c.r. $= r = -\dfrac{1}{2}$

$$t_n = ar^{n-1} = 10 \times \left(-\frac{1}{2}\right)^{n-1} = -.078125 = 10 \times -.0078125$$

$$= 10 \times -\frac{78125}{10^7} = 10 \times \frac{-5^7}{10^7} = 10 \times \left(-\frac{1}{2}\right)^7$$

$t_8 = -.078125 \Rightarrow 10 - 5 + 2.5 - 1.25 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots - .078125$

$$= S_8 = 10 \times \frac{(-1/2)^8 - 1}{(-1/2) - 1} = 10 \times \frac{2}{3} \times \frac{2^8 - 1}{2^8} = \frac{5}{3} \times \frac{2^8 - 1}{2^6}$$

(x) $t_1 = a = r^{10}$, c.r. $= \dfrac{t}{r^3}$,

$$t_n = r^{10} \times \left(\frac{t}{r^3}\right)^{n-1} = \frac{t^{n-1}}{r^{3n-3-10}} = \frac{t^9}{r^{17}} = t_{10}$$

(ii) Let a be the first term of the AP. Its common difference is $a+12$. Hence the AP is $a, a+a+12, a+a+12+a+12$ i.e. $a, 2a+12, 3a+24$

We are given that $a+1, 2a+14, 3a+25$ are in GP. Hence

$(2a+14)^2 = (3a+25)(a+1)$

$\Rightarrow 196+56a+4a^2 = 25+28a+3a^2$

$\Rightarrow a^2+28a+171 = 0 = (a+19)(a+9)$

$\Rightarrow a = -19$ or $a = -9$

The AP is $-19, -26, -33$; or $-9, -6, -3$

The corresponding GP is $-18, -24, -32$; or $-8, -4, -2$

Ex 13. (i) Show that $9^{1/3}.9^{1/9}.9^{1/27}.\ldots\ldots = 3$

(ii) If $S_1, S_2, S_3, \ldots\ldots, S_p$ be the sum of p infinite geometric series whose first terms are $1, 2, 3, \ldots, p$ and whose common ratios are

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \frac{1}{p+1}$ respectively; prove that

$$S_1+S_2+S_3+\ldots\ldots+S_p = \frac{p}{2}(p+3).$$

Soln: (i) $9^{1/3}.9^{1/9}.9^{1/27}.\ldots\ldots = 9^{(1/3+1/9+1/27+\ldots\ldots)}$

We see $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\ldots$

$$=\frac{1}{3}\left\{1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\ldots\ldots\right\}$$

$$=\frac{1}{3}\cdot\frac{1}{1-1/3}=\frac{1}{2}$$

$\Rightarrow 9^{(1/3+1/9+1/27+\ldots\ldots)} = 9^{1/2} = 3$ **Proved.**

(ii) Given $S_1 = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\ldots = \frac{1}{1-1/2} = 2$

$$S_2 = 2\left\{1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\ldots\ldots\right\} = 2.\frac{1}{1-1/3} = 3$$

$$S_3 = 3\left\{1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\ldots\ldots\right\} = 3.\frac{1}{1-1/4} = 4$$

By symmetry $S_p = p+1$

Now $S_1+S_2+S_3+\ldots\ldots+S_p = 2+3+4+\ldots\ldots+(p+1)$

$$=\frac{p}{2}(2+p+1) = \frac{p}{2}(p+3)$$

Ex 14. Find the sum:

(i) $1+3+9+27+\ldots\ldots$ to 11^{th} term.

(ii) $2+4+8+16+\ldots\ldots$ to 8^{th} term.

(iii) $3+\frac{1}{3}+\frac{1}{27}+\ldots\ldots$ to 6^{th} term.

(iv) $2-6+18-54+\ldots\ldots$ to 15^{th}

(v) $2-\frac{3}{4}+\frac{9}{32}+\ldots\ldots$ to 10^{th} term.

(vi) $1+\frac{1}{4}+\frac{1}{16}+\ldots\ldots$ to infinity.

(vii) $9-4+\frac{16}{9}-\ldots\ldots$ to infinity.

(viii) $6+3+1.5+.75+\ldots\ldots+.09375$

(ix) $10-5+2.5-1.25+\ldots\ldots-.078125$

(x) $r^{10}+r^7t+r^4t^2+\ldots\ldots+\frac{t^9}{r^{17}}$

(xi) $4+44+444+\ldots\ldots$ to n terms.

(xii) **[2016]** $7+77+777+\ldots\ldots$ to n terms.

(xiii) $.9+.99+.999+\ldots\ldots$ to n terms.

(xiv) $1.2.5+2.3.6+3.4.7+\ldots\ldots$ to n terms.

(xv) $1.4+2.9+3.16+\ldots\ldots$ to n terms.

(xvi) $3+5+9+15+23+\ldots\ldots$ to n terms.

$a = 576,\quad t_7 = 9 = ar^6 = 576\,r^6 \qquad \Rightarrow r^6 = \dfrac{1}{64} \;\Rightarrow r = \dfrac{1}{2}$

Hence the 5 GMs are 288, 144, 72, 36, 18.

(iii) : If M_1, M_2, M_3, M_4 are 4 GM s, then $4, M_1, M_2, M_3, M_4, \dfrac{1}{256}$ are in GP.

$\Rightarrow t_1 = 4,\ t_6 = \dfrac{1}{256} = 4 \times r^5 = \dfrac{1}{4^4}$

$\Rightarrow r = \dfrac{1}{4} \Rightarrow M_1 = 1,\ M_2 = \dfrac{1}{4},\ M_3 = \dfrac{1}{16},\ M_4 = \dfrac{1}{64}$

Ex 8. [QP 2012, 2014, 2015] If a, b, c are in AP and x, y, z are in GP then prove that $x^{b-c}y^{c-a}z^{a-b} = 1$

Soln: Given a, b, c are in AP $\therefore b - a = c - b \;\Rightarrow a - b = b - c$ ....(I)

Since x, y, z are in GP therefore $y^2 = xz$ ....(II)

Now LHS $= x^{b-c}y^{c-a}z^{a-b} = (xz)^{b-c}y^{c-a}$ from (I)

$= (y^2)^{b-c} \times y^{c-a}$ from (II)

$= y^{2b-2c+c-a}$

$= y^{(b-a)-(c-b)}$

$= y^0 = 1 = \text{RHS}$

Ex 9. If a, b, c are p^{th}, q^{th}, r^{th} term of a GP respectively then prove that

(i) **[QP 2010]** $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$

(ii) **[QP 2011]** $a^{q-r}b^{r-p}c^{p-q} = 1$

Soln: Let x be the first term and s be the common ratio in the GP.

(i) We have $t_1 = x,\ t_p = xs^{p-1} = a,\ t_q = xs^{q-1} = b,\ t_r = xs^{r-1} = c$

$a^{q-r}b^{r-p}c^{p-q} = (xs^{p-1})^{q-r}(xs^{q-1})^{r-p}(xs^{r-1})^{p-q}$

$= (x^{q-r+r-p+p-q})(s^{(pq-pr+qr-pq+rp-qr)+(r-q+p-r+q-p)})$

$= x^0 s^0 = 1$

Taking log on both sides

$\log(a^{q-r}b^{r-p}c^{p-q}) = \log 1 = 0$

$\Rightarrow (q-r)\log a + (r-p)\log b + (p-q)\log c = 0$ Hence proved.

(ii) LHS $= a^{q-r}b^{r-p}c^{p-q} = (xs^{p-1})^{q-r}(xs^{q-1})^{r-p}(xs^{r-1})^{p-q}$

$= (x^{q-r+r-p+p-q})(s^{(pq-pr+qr-pq+rp-qr)+(r-q+p-r+q-p)})$

$= x^0 s^0 = 1 = \text{RHS}.$

Ex 10. If a, b, c, d are in GP then prove that $a+b, b+c, c+d$ are also in GP.

Soln: a, b, c, d are in GP $\Rightarrow b = ar, c = ar^2, d = ar^3$

Now $a + b = a + ar = a(1+r)$

$b + c = ar + ar^2 = ar(1+r) = \{a(1+r)\}r = (a+b)r$

$c + d = ar^2(1+r) = \{a(1+r)\}r^2 = (a+b)\,r^2 = (b+c)r$

So we have $b + c : a + b = c + d : b + c$

$\Rightarrow a+b,\ b+c,\ c+d$ are in GP.

Ex 11. If $\dfrac{1}{x+y}, \dfrac{1}{2y}, \dfrac{1}{y+z}$ are in AP prove that x, y, z are in GP.

Soln: : $\dfrac{1}{x+y}, \dfrac{1}{2y}, \dfrac{1}{y+z}$ are in AP $\Rightarrow \dfrac{1}{2y} - \dfrac{1}{x+y} = \dfrac{1}{y+z} - \dfrac{1}{2y}$

$\Rightarrow \dfrac{x-y}{2xy+2y^2} = \dfrac{y-z}{2yz+2y^2}$

$\Rightarrow (2yz + 2y^2)(x-y) = (2xy + 2y^2)(y-z)$

$\Rightarrow 2xyz + 2xy^2 - 2y^2z - 2y^3 = 2xy^2 + 2y^3 - 2xyz - 2y^2z$

$\Rightarrow 4xyz = 4y^3 \quad \Rightarrow y^2 = xz$

Thus x, y, z are in GP.

Ex 12. (i) What common quantity should be added to each of the numbers 1, 3 and 7 J & o that the new numbers form a GP?

(ii) In an AP the common difference exceeds the first term by 12, the first term increased by 1, second term increased by 2, and the third term increased by 1 are in GP. Find the series.

Soln: (i) Let d be the common value to satisfy $(3+d)^2 = (1+d)(7+d)$ so that $1+d,\ 3+d,\ 7+d$ are in GP.

So $(3+d)^2 = (1+d)(7+d)$

$\Rightarrow 9 + 6d + d^2 = 7 + 8d + d^2 \Rightarrow d = 1$ The resulting GP is 2, 4, 8

$$= 2\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\right) + 3\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\right)$$

$$= 2e + 3e = 5e = \text{RHS}$$

(ii) Here (i) $t_n = \dfrac{(2n-1)}{(n-1)!} = \dfrac{2(n-1)}{(n-1)!}+\dfrac{1}{(n-1)!} = \dfrac{2}{(n-2)!}+\dfrac{1}{(n-1)!}$

Hence LHS $= 1+\dfrac{3}{1!}+\dfrac{5}{2!}+\dfrac{7}{3!}+\dfrac{9}{4!}+\ldots\ldots$

$$= 1+\left(2+\frac{1}{1!}\right)+\left(\frac{2}{1!}+\frac{1}{2!}\right)+\left(\frac{2}{2!}+\frac{1}{3!}\right)+\ldots\ldots$$

$$= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots+2\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\right)$$

$$= e + 2e = 3e = \text{RHS}$$

Ex 4. Prove that: $e = \dfrac{2}{1!}+\dfrac{4}{3!}+\dfrac{6}{5!}+\dfrac{8}{7!}+\ldots\ldots$

Soln: RHS $= \dfrac{2}{1!}+\dfrac{4}{3!}+\dfrac{6}{5!}+\dfrac{8}{7!}+\ldots\ldots$

$$= \frac{1+1}{1!}+\frac{3+1}{3!}+\frac{5+1}{5!}+\frac{7+1}{7!}+\ldots\ldots$$

$$= 1+\frac{1}{1!}+\frac{3}{3!}+\frac{1}{3!}+\frac{5}{5!}+\frac{1}{5!}+\frac{7}{7!}+\frac{1}{7!}+\ldots\ldots$$

$$= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\ldots$$

$$= e = \text{LHS}$$

Ex 5. Prove that: $\dfrac{1}{2!}+\dfrac{1+2}{3!}+\dfrac{1+2+3}{4!}+\dfrac{1+2+3+4}{5!}+\ldots\ldots = \dfrac{e}{2}$

Soln: Here $t_n = \dfrac{1+2+3+\ldots+n}{(n+1)!} = \dfrac{1}{(n+1)!}\cdot\dfrac{n(n+1)}{2} = \dfrac{1}{2(n-1)!}$

Now LHS $= \dfrac{1}{2!}+\dfrac{1+2}{3!}+\dfrac{1+2+3}{4!}+\dfrac{1+2+3+4}{5!}+\ldots\ldots$

$$= \frac{1}{2!}+\frac{1}{2.1!}+\frac{1}{2.2!}+\frac{1}{2.3!}+\ldots\ldots$$

$$= \frac{1}{2}\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\ldots\right) = \frac{e}{2} = \text{RHS}$$

Ex 6. Prove that: $1+\dfrac{1+2}{2!}+\dfrac{1+2+2^2}{3!}+\dfrac{1+2+2^2+2^3}{4!}+\ldots\ldots = e(e-1)$

Soln: Here $t_n = \dfrac{1+2+2^2+\ldots\ldots+2^{n-1}}{n!} = \dfrac{1}{n!}(1+2+2^2+\ldots\ldots+2^{n-1})$

$$= \frac{1}{n!}\,\frac{2^n-1}{2-1} \text{ (sum of a GP)}$$

$$= \frac{2^n-1}{n!}$$

Now LHS $= 1+\dfrac{1+2}{2!}+\dfrac{1+2+2^2}{3!}+\dfrac{1+2+2^2+2^3}{4!}+\ldots\ldots$

$$= \frac{2-1}{1!}+\frac{2^2-1}{2!}+\frac{2^3-1}{3!}+\frac{2^4-1}{4!}+\ldots\ldots$$

$$= \left(\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\ldots\ldots\right)-\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots\right)$$

$$= \left(1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\ldots\ldots\right)-\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots\right)$$

$$= e^2 - e = e(e-1) = \text{RHS}$$

Ex 7. Prove that: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots\ldots = 1$

Soln: LHS $= \dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots\ldots = \dfrac{2-1}{2!}+\dfrac{3-1}{3!}+\dfrac{4-1}{4!}+\ldots\ldots$

Ex 2. (i) **[QP 2013]** Prove that: $1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots = \frac{1}{2}\left(e-\frac{1}{e}\right)$

(ii) **[QP 2013, 2016]** Prove that: $\frac{1.3}{1!}+\frac{2.4}{2!}+\frac{3.5}{3!}+\frac{4.6}{4!}+\ldots\ldots = 4e$

(iii) **[QP 2013, 2015, 2016]** Prove that: $\frac{e-1}{e+1}=\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots}$

Soln: (i) $\text{RHS}=\frac{1}{2}\left(e-\frac{1}{e}\right)=\frac{1}{2}\{(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots)$

$-(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots)\}$

$=\frac{1}{2}\{2(1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots)\}$

$=1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots=\text{LHS}$

(ii) Here $t_n=\frac{n(n+2)}{n!}=\frac{n(n-1)}{n!}+\frac{3n}{n!}=\frac{1}{(n-2)!}+\frac{3}{(n-1)!}$ Hence

$\text{LHS}=\frac{1.3}{1!}+\frac{2.4}{2!}+\frac{3.5}{3!}+\frac{4.6}{4!}+\ldots\ldots=3+(\frac{1}{0!}+\frac{3}{1!})+(\frac{1}{1!}+\frac{3}{2!})+(\frac{1}{2!}+\frac{3}{3!})+\ldots\ldots$

$=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots)+3(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots)$

$=e+3e=4e=\text{RHS}$

(iii) We have $e+e^{-1}=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots)$

$+(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots)$

or, $\frac{1}{e}(e^2+1)=2(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots$

or, $\frac{1}{2e}(e^2+1)-1=\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots$

or, $\frac{1}{2e}(e^2-2e+1)=\frac{1}{2e}(e-1)^2=\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots$ (I)

And $e-e^{-1}=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots)$

$-(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots)$

or, $\frac{1}{e}(e^2-1)=2(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots)$

or, $\frac{1}{2e}(e^2-1)=\frac{1}{2e}(e+1)(e-1)=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots$ (II)

$\therefore \frac{e-1}{e+1}=\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots}$ Hence proved.

Ex 3. (i) Prove that: $3+\frac{5}{1!}+\frac{7}{2!}+\frac{9}{3!}+\frac{11}{4!}+\ldots\ldots=5e$

(ii) **[QP 2015]** Prove that: $1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+\frac{9}{4!}+\ldots\ldots=3e$

Soln: Here (i) $t_n=\frac{(2n+1)}{(n-1)!}=\frac{2(n-1)}{(n-1)!}+\frac{3}{(n-1)!}=\frac{2}{(n-2)!}+\frac{3}{(n-1)!}$

Hence $\text{LHS}=3+\frac{5}{1!}+\frac{7}{2!}+\frac{9}{3!}+\frac{11}{4!}+\ldots\ldots$

$=3+(2+\frac{3}{1!})+(\frac{2}{1!}+\frac{3}{2!})+(\frac{2}{2!}+\frac{3}{3!})+\ldots\ldots$

1.6.10 The number e :

The series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\infty$ is denoted by the letter e.

This number is of fundamental importance in higher mathematics. It is the base of natural logarithm introduced by John Napier. Natural logarithm defines exponential function.

Exponential function: The function $f(x)=e^x$ is known as *exponential function*.

The series derived for e^x is **Exponential series** and is given by

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\infty$$

1.6.11 Exponential Theorem: Exponential Theorem states that

For $a>0$ $\quad a^x=1+x\log_e a+\frac{x^2(\log_e a)^2}{2!}+\frac{x^3(\log_e a)^3}{3!}+\ldots\ldots\infty$

Proof: We have $\quad e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\infty$

for $a>0$ we have $\log_e a=k$ or $a=e^k$

Hence $a^x=e^{kx}=1+\frac{kx}{1!}+\frac{(kx)^2}{2!}+\frac{(kx)^3}{3!}+\ldots\ldots\infty$

$$=1+\frac{kx}{1!}+\frac{k^2x^2}{2!}+\frac{k^3x^3}{3!}+\ldots\ldots\infty$$

$$=1+x\log_e a+\frac{x^2(\log_e a)^2}{2!}+\frac{x^3(\log_e a)^3}{3!}+\ldots\ldots\infty$$

Corollary 1. $e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\ldots\infty$

$$e^{-1}=\frac{1}{e}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots\infty$$

Corollary 2. $a=1+\log_e a+\frac{(\log_e a)^2}{2!}+\frac{(\log_e a)^3}{3!}+\ldots\ldots$

$$a^{-1}=\frac{1}{a}=1-\log_e a+\frac{(\log_e a)^2}{2!}-\frac{(\log_e a)^3}{3!}+\ldots\ldots$$

Worked out examples:

Ex 1.[QP 2012] Prove that $\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots$

Soln: $e^x+e^{-x}=(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots)$

$$+(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\ldots)$$

$$=2(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots)$$

$$\Rightarrow \frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\ldots$$

Ex 2.[QP 2009] Prove that $\frac{e+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots$

Soln: $e+e^{-1}=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots)$

$$+(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots)$$

$$=2(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots)$$

$$\Rightarrow \frac{e+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots$$

Exercise 1.6(B)

1. Find (i) 6^{th} term of $2, 8, 32, \ldots\ldots$

   (ii) 12^{th} term of $4, -16, 64, \ldots\ldots$

   (iii) 8^{th} term in $5, 3, \frac{9}{5}, \ldots\ldots$

Ans: (i) 2048 (ii) -4×4^{11} (iii) $5 \times \left(\frac{3}{5}\right)^7$

2. (i) The 4^{th} term of a GP is $\frac{1}{12}$ and 8^{th} term is $\frac{1}{192}$. Find the 14^{th} term.

   (ii) The 9^{th} term of a GP is 64 and 15^{th} term is 4096. Find the 11^{th} term.

   (iii) The 5^{th} term of a GP is 48 and 12^{th} term is 6144. Find the 1^{st} term and the common ratio.

   (iv) The first two terms in a GP are 3 and 1. Find the 10^{th} and n^{th} term.

   (v) The 5^{th} term of a GP is 16 and 10^{th} term is $\frac{1}{2}$. Find the GP.

Ans: (i) $\frac{1}{12288}$ (ii) 256 (iii) 1^{st} term is 3, common ratio is 2

(iv) $t_{10} = \frac{1}{6561}$, $t_n = \frac{1}{3^{n-2}}$ (v) $256, 128, 64, \ldots\ldots$

3. Insert (i) 3 GM between $\frac{1}{3}$ and 432.

   (ii) 3 GM between 1 and 256.

   (iii) 6 GM between 27 and $\frac{1}{81}$

   (iv) 3 GM between a and $\frac{1}{a}$

   (v) 3 GM between 15 and $\frac{5}{27}$

Ans: (i) 2, 12, 72 (ii) 4, 16, 64 (iii) $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

(iv) $\sqrt{a}, 1, \frac{1}{\sqrt{a}}$ (v) $5, \frac{5}{3}, \frac{5}{9}$

4. If $5, x, y, z, 405$ are the first five terms of a GP then find the value of x, y, z.

   Ans: 15, 45, 135 or, $-15, 45, -135$

5. The sum of three terms in GP is $\frac{7}{6}$ and product of these terms is $\frac{1}{27}$. Find the terms.

   Ans: $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}$

6. The sum of three consecutive terms in a GP is 21 and the sum of their squares is 189. Find the three terms.

   Ans: 3, 6, 12

7. If x, y, z are in GP then prove that $\log x, \log y, \log z$ are in AP.
8. The AM between a and b is 15 and their geometric mean is 9. Find a, b.

   Ans: $a = 27, b = 3$.

9. Find the three terms in a GP whose product is 729 and the sum of their product in pairs is 819.

   Ans: 1, 9, 81

10. Find the sum:

    (i) $1 + 2 + 4 + \ldots\ldots + 2^9$

    (ii) $\frac{1}{\sqrt{2}} - 1 + \sqrt{2} - \ldots\ldots - 16$

    (iii) $5 + 55 + 555 + \ldots\ldots$ to n terms.

    (iv) $2^2 + 4^2 + 6^2 + \ldots\ldots + 16^2$

    (v) $\frac{4}{3} - 1 + \frac{3}{4} - \frac{9}{16} + \ldots\ldots$

Ans: (i) 1023 (ii) $-\frac{31}{2}(2-\sqrt{2})$ (iii) $\frac{50}{81}(10^n - 1)$ (iv) 816 (v) $\frac{16}{21}$

$$=(\frac{1}{2}-\frac{1}{3})+(\frac{1}{4}-\frac{1}{5})+(\frac{1}{6}-\frac{1}{7})+\dots\dots$$

$$=1-(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots\dots)$$

$$=1-\log_e 2$$

$=$ RHS Proved

(ii) LHS $=\frac{1}{2}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\dots\dots$

$$=\frac{1}{2}+\frac{1}{2}(\frac{1}{1.2}-\frac{1}{2.3})+\frac{1}{2}(\frac{1}{3.4}-\frac{1}{4.5})+\frac{1}{2}(\frac{1}{5.6}-\frac{1}{6.7})+\dots\dots$$

$$=\frac{1}{2}[1+\{(1-\frac{1}{2})-(\frac{1}{2}-\frac{1}{3})\}+\{(\frac{1}{3}-\frac{1}{4})-(\frac{1}{4}-\frac{1}{5})\}$$
$$+\{(\frac{1}{5}-\frac{1}{6})-(\frac{1}{6}-\frac{1}{7})\}+\dots\dots]$$

$$=\frac{1}{2}\times 2(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots\dots)$$

$=\log_e 2=$ RHS Proved

(iii) LHS $=\frac{1}{1.2}+\frac{1}{2.2^2}+\frac{1}{3.2^3}+\dots\dots$

$$=-(-\frac{1}{2}-\frac{(\frac{1}{2})^2}{2}-\frac{(\frac{1}{2})^3}{3}-\dots\dots)$$

$=-\log_e(1-\frac{1}{2})=-\log_e\frac{1}{2}=\log_e 2=$ RHS Proved

Ex 3. Find the coefficient of x^9 in the expansion of $\log_e(1+x+x^2)$

Soln: $\log_e(1+x+x^2)=\log_e\frac{1-x^3}{1-x}=\log_e(1-x^3)-\log_e(1-x)$

$$=(-x^3-\frac{x^6}{2}-\frac{x^9}{3}-\dots\dots)-(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\dots)$$

coefficient of x^9 in the expansion is $-\frac{1}{3}+\frac{1}{9}=-\frac{2}{9}$

Ex 4. Find the sum to infinity: $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\dots\dots$

Soln: $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\dots\dots$

$$=\frac{3+2}{1.2.3}+\frac{3+4}{3.4.5}+\frac{3+6}{5.6.7}+\dots\dots$$

$$=3\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\dots\dots\}+\{\frac{2}{1.2.3}+\frac{4}{3.4.5}+\frac{6}{5.6.7}+\dots\dots\}$$

$$=3\{\frac{1}{2}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\dots\dots\}-\frac{3}{2}$$
$$+\frac{1}{2}\{(\frac{1}{1.2}+\frac{1}{2.3})+(\frac{1}{3.4}+\frac{1}{4.5})+(\frac{1}{5.6}+\frac{1}{6.7})+\dots\dots\}$$

$$=3\log_e 2-\frac{3}{2}+\frac{1}{2}[\{(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})\}+\{(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})\}$$
$$+\{(\frac{1}{5}-\frac{1}{6})+(\frac{1}{6}-\frac{1}{7})\}+\dots\dots]$$

$$=3\log_e 2-\frac{3}{2}+\frac{1}{2}\times 1$$

$$=3\log_e 2-1$$

$$\Rightarrow \log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots\ldots\ldots\right)$$

$$\Rightarrow \frac{1}{2}\log_e \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots\ldots\ldots$$

$$\Rightarrow \log_e \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots\ldots\ldots \quad (\text{for } -1 < x < 1)$$

(c) $\log_e(1+x) + \log_e(1-x)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots + (-1)^{r-1}\frac{x^r}{r} + \ldots\ldots$$

$$+ \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots - \frac{x^r}{r} - \ldots\ldots\right)$$

$$= -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\ldots\right)$$

$$\Rightarrow \log_e(1-x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\ldots\right)$$

$$\Rightarrow -\frac{1}{2}\log_e(1-x^2) = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\ldots \quad (\text{for } -1 < x < 1)$$

Worked out examples:

Ex 1.(i) [QP 2012] Prove that: $\log_e \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots\ldots$

(ii) **[QP 2014]** Prove that: $\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} \ldots\ldots\ldots\right)$

Soln: (i) LHS $= \log_e \sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log_e \frac{1+x}{1-x}$

$$= \frac{1}{2}\{\log_e(1+x) - \log_e(1-x)\}$$

$$= \frac{1}{2} \times \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots + (-1)^{r-1}\frac{x^r}{r} + \ldots\ldots\right.$$

$$\left. - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots - \frac{x^r}{r} - \ldots\ldots\right)\right\}$$

$$= \frac{1}{2} \times 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\right)$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots = \text{RHS} \quad \text{Proved}$$

(ii) LHS $= \log_e \frac{1+x}{1-x} = \log_e(1+x) - \log_e(1-x)$

$$= \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots + (-1)^{r-1}\frac{x^r}{r} + \ldots\ldots\right.$$

$$\left. - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots - \frac{x^r}{r} - \ldots\ldots\right)\right\}$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\ldots\right) = \text{RHS} \quad \text{Proved}$$

Ex 2. Prove that: (i) $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots\ldots\ldots = 1 - \log_e 2$

(ii) $\frac{1}{2} + \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \ldots\ldots\ldots = \log_e 2$

(iii) $\frac{1}{1.2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \ldots\ldots\ldots = \log_e 2$

Soln: (i) LHS $= \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots\ldots\ldots$

Exercise 1.6(C)

1. Prove that $\frac{2}{1!}+\frac{4}{2!}+\frac{6}{3!}+\frac{8}{4!}+\ldots\ldots\ldots\infty=2e$

2. Prove that $\frac{1}{1!}+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots\ldots\ldots\infty=e+1$

3. Prove that $\frac{1.2}{1!}+\frac{2.3}{2!}+\frac{3.4}{3!}+\frac{4.5}{4!}+\ldots\ldots\ldots\infty=3e$

4. Prove that $\frac{1^2}{1!}+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\ldots\ldots\infty=2e$

5. Prove that $1+\frac{1}{2!}+\frac{1.3}{4!}+\frac{1.3.5}{6!}+\ldots\ldots\ldots\infty=\sqrt{e}$

6. Prove that $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots\ldots\ldots\infty=\frac{3e}{2}$

7. Prove that $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{8}{9!}+\ldots\ldots\ldots\infty=\frac{1}{e}$

8. Prove that $\frac{1}{1.3}+\frac{1}{1.2.3.5}+\frac{1}{1.2.3.4.5.7}+\ldots\ldots\ldots\infty=\frac{1}{e}$

9. Prove that $\frac{1^2}{2!}+\frac{2^2}{3!}+\frac{3^2}{4!}+\frac{4^2}{5!}+\ldots\ldots\ldots\infty=e-1$

10. Prove that $1+\frac{1+3}{2!}+\frac{1+3+3^2}{3!}+\ldots\ldots\ldots\infty=\frac{e}{2}(e^2-1)$

11. Prove that $\frac{2.5}{1!}+\frac{3.6}{2!}+\frac{4.7}{3!}+\ldots\ldots\ldots\infty=11e-4$

12. Find the sum of $1+\frac{1+3}{2!}x+\frac{1+3+5}{3!}x^2+\frac{1+3+5+7}{4!}x^3+\ldots\ldots\infty$

Ans: $(x+1)e^x$

13. Find the sum of $\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+\frac{4}{9!}+\ldots\ldots\ldots\infty$

Ans: $\frac{1}{2e}$

14. In the expansion of $\frac{1-x+x^2}{e^x}$ find the coefficient of x^3.

Ans: $-\frac{5}{3}$

* * *

1.6.12. LOGARITHMIC SERIES

If $-1<x<1$ then

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots\ldots+(-1)^{r-1}\frac{x^r}{r}+\ldots\ldots\ldots$$

This series is known as Logarithmic series.

Note: Replacing x by $-x$ we have

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\ldots\ldots-\frac{x^r}{r}+\ldots\ldots\ldots$$

1.6.13 Important Deduction:

(a) The theorem remains true for $x=1$. Thus

$$\log_e(1+1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots+(-1)^{r-1}\frac{1}{r}+\ldots\ldots\ldots$$

$$\Rightarrow \log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots$$

(b) $\log_e(1+x)-\log_e(1-x)$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots\ldots+(-1)^{r-1}\frac{x^r}{r}+\ldots\ldots$$

$$-(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\ldots\ldots-\frac{x^r}{r}+\ldots\ldots)$$

$$=2(x+\frac{x^3}{3}+\frac{x^5}{5}\ldots\ldots\ldots)$$

$$=(1-\frac{1}{2!})+(\frac{1}{2!}-\frac{1}{3!})+(\frac{1}{3!}-\frac{1}{4!})+\ldots\ldots\ldots\ldots$$

$$=1-\frac{1}{2!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{3!}-\frac{1}{4!}+\ldots\ldots\ldots\ldots=1=\text{RHS}$$

Ex 8. Find the value of $\frac{1}{\sqrt{e}}$ up to 3 decimal places.

Soln: $\frac{1}{\sqrt{e}}=e^{-1/2}=1-\frac{\frac{1}{2}}{1!}+\frac{(\frac{1}{2})^2}{2!}-\frac{(\frac{1}{2})^3}{3!}+\frac{(\frac{1}{2})^4}{4!}-\ldots\ldots\ldots$

$$=1-\frac{1}{2}+\frac{1}{8}-\frac{1}{48}+\frac{1}{384}-\ldots\ldots\ldots$$

$$=1-.5+.125-.021+.003+\ldots\ldots=0.607$$

Ex 9. [QP 2010] Prove that:

$$(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots)(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\ldots\ldots\ldots)=1$$

Soln: LHS $=(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots\ldots)(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\ldots\ldots\ldots)$

$$=e^x\times e^{-x}=e^0=1=\text{RHS}$$

Ex 10. Find the sum of :

(i) $\frac{2}{1!}+\frac{2+4}{2!}+\frac{2+4+6}{3!}+\frac{2+4+6+8}{4!}+\ldots\ldots\ldots\ldots$

(ii) $1-log_e 2+\frac{(\log_e 2)^2}{2!}-\frac{(\log_e 2)^3}{3!}+\ldots\ldots\ldots$

(iii) $(1+3)\,log_e 3+\frac{1+3^2}{2!}(log_e 3)^2+\frac{1+3^3}{3!}(log_e 3)^3+\ldots\ldots\ldots$

Soln: (i) Here $t_n=\frac{2(1+2+3+\ldots\ldots+n)}{n!}=\frac{1}{n!}\times 2\times\frac{n(n+1)}{2}$

$$=\frac{n(n+1)}{n!}=\frac{n(n-1)+2n}{n!}=\frac{n(n-1)}{n!}+\frac{2n}{n!}$$

$$=\frac{1}{(n-2)!}+\frac{2}{(n-1)!}$$

Now $\frac{2}{1!}+\frac{2+4}{2!}+\frac{2+4+6}{3!}+\frac{2+4+6+8}{4!}+\ldots\ldots\ldots\ldots$

$$=2+(1+\frac{2}{1!})+(\frac{1}{1!}+\frac{2}{2!})+(\frac{1}{2!}+\frac{2}{3!})+\ldots\ldots\ldots\ldots$$

$$=2(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots\ldots\ldots)+(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots\ldots\ldots\ldots)$$

$$=2e+e=3e$$

(ii) From Corollary 2 writing $a=2$

$$1-log_e 2+\frac{(\log_e 2)^2}{2!}-\frac{(\log_e 2)^3}{3!}+\ldots\ldots\ldots=\frac{1}{2}$$

(iii) $(1+3)\,log_e 3+\frac{1+3^2}{2!}(log_e 3)^2+\frac{1+3^3}{3!}(log_e 3)^3+\ldots\ldots\ldots$

$$=(\,log_e 3+\frac{(\log_e 3)^2}{2!}+\frac{(\log_e 3)^3}{3!}+\ldots\ldots\ldots)$$

$$+(3log_e 3+\frac{3^2(\log_e 3)^2}{2!}+\frac{3^3(\log_e 3)^3}{3!}+\ldots\ldots\ldots)$$

$$=-2\;+(1+3log_e 3+\frac{3^2(\log_e 3)^2}{2!}+\frac{3^3(\log_e 3)^3}{3!}+\ldots\ldots\ldots)$$

$$+(1+3log_e 3+\frac{3^2(\log_e 3)^2}{2!}+\frac{3^3(\log_e 3)^3}{3!}+\ldots\ldots\ldots)$$

$$=-2+3+3^3$$

$$=28$$

(ii) [QP 2009] Evaluate: $\begin{vmatrix} 99 & 81 & 321 \\ 88 & 72 & 423 \\ 55 & 45 & 657 \end{vmatrix}$

(iii) [QP 2011] Find the value of: $\begin{vmatrix} 23 & 13 & 4 \\ 42 & 39 & 12 \\ 67 & 52 & 16 \end{vmatrix}$

Soln: (i) Minor of a_{31} is $\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 1.2 - 0.3 = 2$

Cofactor of a_{31} is $(-1)^{3+1}\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 2$

Minor of a_{22} is $\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = 2(-2) - 3.4 = -16$

Cofactor of a_{22} is $(-1)^{2+2}\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -16$

(ii) $\begin{vmatrix} 99 & 81 & 321 \\ 88 & 72 & 423 \\ 55 & 45 & 657 \end{vmatrix} = 11\times 9\begin{vmatrix} 9 & 9 & 321 \\ 8 & 8 & 423 \\ 5 & 5 & 657 \end{vmatrix} = 99\times 0 = 0$ since $C_1 = C_2$

(iii) $\begin{vmatrix} 23 & 13 & 4 \\ 42 & 39 & 12 \\ 67 & 52 & 16 \end{vmatrix} = 13\times 4\times\begin{vmatrix} 23 & 1 & 1 \\ 42 & 3 & 3 \\ 67 & 4 & 4 \end{vmatrix} = 52\times 0 = 0$ since $C_2 = C_3$

Ex 2. Prove that

(i) **[QP 2001, 10, 11, 15, 16]** $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + ab + bc + ca$

(iii) $\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ a^3 & b^3 & 1 \end{vmatrix} = ab(a-1)(b-1)(b-a)$

(iv) **[QP 2014]** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = ab$

(v) **[QP 2015]** $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix} = 0$

Soln: (i) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= abc\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$ $C_1 - C_2 \to C_1$, $C_2 - C_3 \to C_2$

$= abc(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$

$= abc(a-b)(b-c)\{(b+c)-(a+b)\}$

$= abc(a-b)(b-c)(c-a)$ Proved.

2. The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

When expanded it is seen that $\Delta_1 = \Delta_2$

3. When two rows (or column) of a determinant are interchanged then its numerical value remains same but sign is changed.

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

When expanded it is seen that $\Delta_1 = -\Delta_2$

4. If each element of a row (or column) of a determinant is multiplied by a constant α then its value gets multiplied by .

i.e. $$\begin{vmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \alpha \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

5. If two rows (or column) of a determinant are identical then the value of the determinant is zero.

i.e. $$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0$$

Theorem 1: If every element in any row (or column) consists of the sum (or difference) of two quantities then the determinant can be expressed as the sum (or difference) of determinants of same order. Thus we have

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 2: The value of a determinant remains unaltered by adding (or subtracting) to all the elements of any particular row (or column) the same multiple of corresponding elements of one more other rows (or columns). This means

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$$

1.7.5 Adjoint of determinant: The determinant obtained by replacing each element of a given determinant by its cofactor is known as adjoint of the determinant.

1.7.6 Cramer's Rule: Consider the linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Here x, y, z are unknowns. we compute the following determinants-

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The condition required is $\Delta \neq 0$.

Solution is given by $x = \Delta_1/\Delta \quad y = \Delta_2/\Delta \quad z = \Delta_3/\Delta$

This is known as Cramer's rule.

Worked out Examples:

Ex 1. (i) Find the minor and cofactor of the element $a_{31} = 4$ and $a_{22} = 0$ in

$$\begin{vmatrix} 2 & 1 & 3 \\ -3 & 0 & 2 \\ 4 & 1 & -2 \end{vmatrix}$$

$(8) \times b_1 \Rightarrow b_1 a_2 x + b_1 b_2 y = 0 \quad \ldots\ldots\ldots\ldots(10)$

$(9) - (10) \Rightarrow (b_2 a_1 - b_1 a_2)\, x = 0$

As $x \neq 0$ hence $b_2 a_1 - b_1 a_2 = 0$

Now let us take three linear equations in three variables x, y, z.

$$a_1 x + b_1 y + c_1 z = 0$$

$$a_2 x + b_2 y + c_2 z = 0$$

$$a_3 x + b_3 y + c_3 z = 0$$

We find a relation from these nine coefficients if the three equations are satisfied by the same value of x, y, z. We get

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{y}{c_2 a_3 - c_3 a_2} = \frac{z}{a_2 b_3 - a_3 b_2}$$

Substituting the proportionate value of x, y, z in the first equation we have

$$a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2) = 0$$

This expression is equivalent to the third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad \text{Or we can write}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2) \ldots\ldots(11)$$

The n^{th} order determinant is written as $\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & a_{nn} \end{vmatrix}$

Its elements are a_{ij} where i is the row number and j is the column number.

1.7.2 Minor and Cofactor : The minor of any element of a determinant is the determinant obtained by omitting the row and the column in which the particular element occurs. The minor of b_1 in the third order determinant (11) is the 2^{nd} order determinant

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

The cofactor of an element is the minor of that element with coefficient $(-1)^{i+j}$ where i is the row number and j is the column number of the element. The cofactor of b_1 in the third order determinant (11) is $(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = B_1$

Cofactor of a_{ij} is usually denoted by A_{ij}.

1.7.3 Laplace's Expansion : It is the expansion of the determinant in terms cofactors. It can be verified that value of determinant (11) is any one of the following expressions:

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1 \quad \Delta = a_2 A_2 + b_2 B_2 + c_2 C_2 \quad \Delta = a_3 A_3 + b_3 B_3 + c_3 C_3$$

$$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 \quad \Delta = b_1 B_1 + b_2 B_2 + b_3 B_3 \quad \Delta = c_1 C_1 + c_2 C_2 + c_3 C_3$$

Thus a determinant can be developed in terms of the elements of any row or any columns. We can say – *The value of a determinant is given by the sum of the products formed by multiplying the elements of any row (or column) by the corresponding cofactors.*

1.7.4 Properties of Determinant: The following properties of determinant are same for any order.

1. If every element of a row (or column) of a determinant is zero then determinant vanishes

$$\Delta = \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0(b_2 c_3 - b_3 c_2) - 0(b_1 c_3 - b_3 c_1) + 0(b_1 c_2 - b_2 c_1) = 0$$

Exercise 1.6(D)

1. Prove that

(i) $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\ldots\ldots\ldots=\log_e 2$

(ii) $\frac{1}{1.3}-\frac{1}{2.3^2}+\frac{1}{3.3^3}-\ldots\ldots\ldots=\log_e \frac{4}{3}$

(iii) $1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\ldots\ldots\ldots=\log_e 3$

(iv) $(\frac{1}{5}+\frac{1}{7})+\frac{1}{3}\{(\frac{1}{5})^3+(\frac{1}{7})^3\}+\frac{1}{5}\{(\frac{1}{5})^5+(\frac{1}{7})^5\}+\ldots\ldots\ldots=\log_e \sqrt{2}$

(v) $\frac{1}{1.2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\ldots\ldots\ldots=\log_e \frac{3}{2}$

2. Find the sum to infinity

(i) $\frac{1}{1.3}+\frac{1}{2}.\frac{1}{3.5}+\frac{1}{3}.\frac{1}{5.7}+\ldots\ldots\ldots$ Ans: $\log_e 2-1$

(ii) $1+(\frac{1}{2}+\frac{1}{3})\frac{1}{4}+(\frac{1}{4}+\frac{1}{5})\frac{1}{4^2}+\ldots\ldots\ldots$ Ans: $\frac{1}{2}\log_e 12$

(iii) $\frac{1}{2.3.4}+\frac{1}{4.5.6}+\frac{1}{6.7.8}+\ldots\ldots\ldots$ Ans: $\frac{3}{4}-\log_e 2$

(iv) $\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+\ldots\ldots\ldots$ Ans: $2-2\log_e 2$

(v) $\frac{1}{4}-\frac{1}{2}.\frac{1}{4^2}+\frac{1}{3}.\frac{1}{4^3}-\frac{1}{4}.\frac{1}{4^4}+\ldots\ldots$ Ans: : $\log_e \frac{5}{4}$

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1.7 DETERMINANT

1.7.1 Formation: First we will discuss how a determinant forms. Let us consider two linear equations-

$$a_1x+b_1y=d_1 \quad \ldots\ldots(1)$$
$$a_2x+b_2y=d_2 \quad \ldots\ldots(2)$$
$$(1)\times b_2 \Rightarrow b_2a_1x+b_2b_1y=b_2d_1 \quad \ldots\ldots\ldots(3)$$
$$(2)\times b_1 \Rightarrow b_1a_2x+b_1b_2y=b_1d_2 \quad \ldots\ldots\ldots(4)$$
$$(3)-(4) \Rightarrow (b_2a_1-b_1a_2)x=b_2d_1-b_1d_2$$
$$\Rightarrow x=\frac{b_2d_1-b_1d_2}{b_2a_1-b_1a_2}$$

Again
$$(1)\times a_2 \Rightarrow a_2a_1x+a_2b_1y=a_2d_1 \quad \ldots\ldots\ldots(5)$$
$$(2)\times a_1 \Rightarrow a_1a_2x+a_1b_2y=a_1d_2 \quad \ldots\ldots\ldots(6)$$
$$(5)-(6) \Rightarrow (a_2b_1-a_1b_2)y=a_2d_1-a_1d_2$$
$$\Rightarrow y=\frac{a_2d_1-a_1d_2}{a_2b_1-a_1b_2}=\frac{a_1d_2-a_2d_1}{b_2a_1-b_1a_2}$$

The common denominator $(b_2a_1-b_1a_2)$ can be written as

$$b_2a_1-b_1a_2=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \ldots\ldots\ldots(\mathrm{I})$$

Right hand side of (I) is called determinant of second order

Thus p,q,r,s are elements of determinant $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ and $\begin{vmatrix} p & q \\ r & s \end{vmatrix}=ps-qr$

In (1) and (2) if d_1 and d_2 equal to 0 then

$$a_1x+b_1y=0 \quad \ldots\ldots(7)$$
$$a_2x+b_2y=0 \quad \ldots\ldots(8)$$
$$(7)\times b_2 \Rightarrow b_2a_1x+b_2b_1y=0 \quad \ldots\ldots\ldots(9)$$

Now $x = \Delta_1 / \Delta = 1/1 = 1$

$y = \Delta_2 / \Delta = 1/1 = 1$

$z = \Delta_3 / \Delta = -1/1 = -1$ Hence solved.

(iv) Here $\Delta = \begin{vmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} 9 & -1 & 0 \\ 7 & 1 & 0 \\ 4 & 1 & 1 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} 5 & 9 & 0 \\ 3 & 7 & 0 \\ 1 & 4 & 1 \end{vmatrix}$ $\quad \Delta_3 = \begin{vmatrix} 5 & -1 & 9 \\ 3 & 1 & 7 \\ 1 & 1 & 4 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0 - 0 + 1(5 + 3) = 8$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 0 \\ 7 & 1 & 0 \\ 4 & 1 & 1 \end{vmatrix} = 0 - 0 + 1(9 + 7) = 16$$

$$\Delta_2 = \begin{vmatrix} 5 & 9 & 0 \\ 3 & 7 & 0 \\ 1 & 4 & 1 \end{vmatrix} = 0 - 0 + 1(35 - 27) = 8$$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 9 \\ 3 & 1 & 7 \\ 1 & 1 & 4 \end{vmatrix} = 5(4 - 7) - (-1)(12 - 7) + 9(3 - 1) = 8$$

Now $x = \Delta_1 / \Delta = 16/8 = 2$

$y = \Delta_2 / \Delta = 8/8 = 1$

$z = \Delta_3 / \Delta = 8/8 = 1$ Hence solved.

(v) Here $\Delta = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ -1 & -2 & 6 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} 5 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & -1 & 6 \end{vmatrix}$ $\quad \Delta_3 = \begin{vmatrix} 5 & -1 & 5 \\ 2 & 3 & 2 \\ 5 & -2 & -1 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix} = 5(18 + 10) - (-1)(12 - 25) + 4(-4 - 15) = 51$$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ -1 & -2 & 6 \end{vmatrix} = 5(18 + 10) - (-1)(12 + 5) + 4(-4 + 3) = 153$$

$$\Delta_2 = \begin{vmatrix} 5 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & -1 & 6 \end{vmatrix} = 5(12 + 5) - 5(12 - 25) + 4(-2 - 10) = 102$$

$$\Delta_3 = \begin{vmatrix} 5 & -1 & 5 \\ 2 & 3 & 2 \\ 5 & -2 & -1 \end{vmatrix} = 5(-3 + 4) - (-1)(-2 - 10) + 5(-4 - 15) = -102$$

Now $x = \Delta_1 / \Delta = 153/51 = 3$

$y = \Delta_2 / \Delta = 102/51 = 2$

$z = \Delta_3 / \Delta = -102/51 = -2$ Hence solved.

(vi) Here $\Delta = \begin{vmatrix} 3 & 1 & -4 \\ 5 & 1 & 3 \\ 1 & -3 & -4 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} 0 & 1 & -4 \\ 1 & 1 & 3 \\ 5 & -3 & -4 \end{vmatrix}$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 1 & -2 & -3 \end{vmatrix} = 2(-3-2) - 1(-6+1) + (-1)(-4-1) = 0$$

$$\Delta_3 = \begin{vmatrix} 2 & 0 & 1 \\ 2 & 4 & 1 \\ 1 & -8 & -2 \end{vmatrix} = 2(-8+8) - 0 + 1(-16-4) = -20$$

Now $x = \Delta_1/\Delta = (-20)/(-20) = 1$
$y = \Delta_2/\Delta = 0/(-20) = 0$
$z = \Delta_3/\Delta = (-20)/(-20) = 1$ Hence solved.

(ii) Here $\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -4 & 5 \\ 7 & -1 & 1 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} 3 & 2 & -1 \\ 7 & -4 & 5 \\ 14 & -1 & 1 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 7 & 5 \\ 7 & 14 & 1 \end{vmatrix}$ $\quad \Delta_3 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -4 & 7 \\ 7 & -1 & 14 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -4 & 5 \\ 7 & -1 & 1 \end{vmatrix} = 1(-4+5) - 2(3-35) + (-1)(-3+28) = 40$$

$$\Delta_1 = \begin{vmatrix} 3 & 2 & -1 \\ 7 & -4 & 5 \\ 14 & -1 & 1 \end{vmatrix} = 3(-4+5) - 2(7-70) + (-1)(-7+56) = 80$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 7 & 5 \\ 7 & 14 & 1 \end{vmatrix} = 1(7-70) - 3(3-35) + (-1)(42-49) = 40$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -4 & 7 \\ 7 & -1 & 14 \end{vmatrix} = 1(-56+7) - 2(42-49) + 3(-3+28) = 40$$

Now $x = \Delta_1/\Delta = 80/40 = 2$
$y = \Delta_2/\Delta = 40/40 = 1$
$z = \Delta_3/\Delta = 40/40 = 1$ Hence solved.

(iii) Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ $\quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix}$ $\quad \Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1(4-1) - 1(2-1) + 1(1-2) = 1$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1(4-1) - 2(2-1) + 0 = 1$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1(4-0) - 1(2-1) + 1(0-2) = 1$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-2) - 1(0-2) + 1(1-2) = -1$$

(ii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$ $C_1+C_2+C_3 \to C_1$

$= 0$ since each element of C_1 is 0. Ans: 0

Ex 4. [QP 2013] Without expansion prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Soln:: LHS $= \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ $C \leftrightarrow R$

$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$ $C_1 \leftrightarrow C_2$

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ $R_1 \leftrightarrow R_2$

= RHS Proved.

4

Ex.5. [QP 2012, 2014] Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Soln: LHS $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$ $C_1-C_2 \to C_1,\ C_2-C_3 \to C_2$

$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$

$= (a-b)(b-c)\{(b+c)-(a+b)\}$

$= (a-b)(b-c)(c-a) =$ RHS Proved.

Ex 6. Solve by Cramer's rule

(i) **[QP 2010]** $2x - z = 1,\ 2x+4y-z = 1,\ x-8y-3z = -2$

(ii) $x+2y-z = 3,\ 3x-4y+5z = 7,\ 7x-y+z = 14$

(iii) **[QP 2010]** $x+y+z = 1,\ x+2y+z = 2,\ x+y+2z = 0$

(iv) **[QP 2011]** $5x-y = 9,\ 3x+y = 7,\ x+y+z = 4$

(v) **[QP 2012, 2014]** $5x-y+4z = 5,\ 2x+3y+5z = 2,\ 5x-2y+6z = -1$

(vi) **[QP 2009]** $3x+y-4z = 0,\ 5x+y+3z = 1,\ x-3y-4z = 5$

(vii) **[QP 2014]** $x+y+z = 3,\ 2x-y+3z = 4,\ x+2y-z = 2$

(viii) **[QP 2015]** $x+2y-z = 2,\ x-y+z = 1,\ x+y+z = 4$

Soln: (i) Here $\Delta = \begin{vmatrix} 2 & 0 & -1 \\ 2 & 4 & -1 \\ 1 & -8 & -3 \end{vmatrix}$ $\Delta_1 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 4 & -1 \\ -2 & -8 & -3 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 1 & -2 & -3 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} 2 & 0 & 1 \\ 2 & 4 & 1 \\ 1 & -8 & -2 \end{vmatrix}$

$\Delta = \begin{vmatrix} 2 & 0 & -1 \\ 2 & 4 & -1 \\ 1 & -8 & -3 \end{vmatrix} = 2(-12-8)-0+(-1)(-16-4) = -20$

$\Delta_1 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 4 & -1 \\ -2 & -8 & -3 \end{vmatrix} = 1(-12-8)-0+(-1)(-8+8) = -20$

(ii) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a & 0 & 1 \\ -b & b & 1 \\ 0 & -c & 1+c \end{vmatrix}$ $C_1 - C_2 \to C_1,\ C_2 - C_3 \to C_2$

$= a\{b(1+c) - (-c)\} + 0 + bc$

$= abc + ab + bc + ca$ Proved.

(iii) $\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & 1 \\ a^3 & b^3 & 1 \end{vmatrix} = ab \begin{vmatrix} 1 & 1 & 1 \\ a & b & 1 \\ a^2 & b^2 & 1 \end{vmatrix}$

$= ab \begin{vmatrix} 0 & 0 & 1 \\ a-1 & b-1 & 1 \\ a^2-1 & b^2-1 & 1 \end{vmatrix}$ $C_1 - C_3 \to C_1,\ C_2 - C_3 \to C_2$

$= ab(a-1)(b-1) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ a+1 & b+1 & 1 \end{vmatrix}$

$= ab(a-1)(b-1)\{(b+1)-(a+1)\} = ab(a-1)(b-1)(b-a)$. Proved.

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ -a & 1+a & 1 \\ 0 & 1 & 1+b \end{vmatrix}$ $C_1 - C_2 \to C_1$

$= 0 - (-a)\{(1+b)-1\} + 0 = ab$. Proved.

(v) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$

$= \begin{vmatrix} 0 & c(b-a) & c(b-a)(a+b+c) \\ 0 & a(c-b) & a(c-b)(a+b+c) \\ 1 & ab & ab(a+b) \end{vmatrix}$ $R_1 - R_2 \to R_1,\ R_2 - R_3 \to R_2$

$= \{c(b-a)\}\{a(c-b)\} \begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & ab & ab(a+b) \end{vmatrix} = 0$ since $R_1 = R_2$

Ex 3. Evaluate:

(i) **[QP 2010]** $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$ (ii) **[2016]** $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

ω is cube root of unity.

Soln: (i) Soln: (i) $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 1-\omega^2 & \omega^2 \\ 0 & 1-\omega & \omega \\ \omega^2-\omega & \omega-1 & 1 \end{vmatrix}$ $C_1 - C_2 \to C_1,\ C_2 - C_3 \to C_2$

$= (\omega-1)^2 \begin{vmatrix} 0 & -1-\omega & \omega^2 \\ 0 & -1 & \omega \\ \omega & \omega & 1 \end{vmatrix} = (\omega-1)^2\omega\{\omega(-\omega-1)+\omega^2\}$

$= (-\omega^2)(\omega^2 - 2\omega + 1) = (-\omega^2)(-3\omega)$ since $\omega^2 + \omega + 1 = 0$

Ans: 3

$= 3\omega^3 = 3$

1.8 MATRICES

Let us consider the system of equation

$$x+y-2z-4=0$$
$$3x-2y-z+1=0$$
$$2x+11y-5z+7=0$$

The coefficients of x, y, z along with the constant terms can be placed in some order. The following arrangement makes it easy to write the system of equation.

$$\begin{bmatrix} 1 & 1 & -2 & -4 \\ 3 & -2 & -1 & 1 \\ 2 & 11 & -5 & 7 \end{bmatrix}$$

This rectangular array is an example of matrix. The horizontal lines are called rows and the vertical lines are called columns. If a matrix A has m rows and n columns, we say A is an 'm by n' matrix and write it as $A_{m\times n}$. It is to be remembered that a matrix has got no numerical value. If we write 5 we understand it to be the number 5, but if we write $[5]$ we will mean it to be the 1×1 matrix, whose only element is 5.

For an $m\times n$ matrix A, number of rows is m and number of columns is n. We write the elements or entries as a_{ij}, i is the row number and j is the column number.

$$A_{6\times 4}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \\ a_{61} & a_{62} & a_{63} & a_{64} \end{bmatrix}$$

Two matrices A and B of same order are **equal** if their corresponding elements are

equal. If row number and column number of a matrix are equal, then the matrix is a **square matrix**. For a square matrix **determinant of the matrix** is defined. In this case we get a value for the determinant. For the 4×4 matrix

$$A=\begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 1 & 5 & -2 \\ -2 & 3 & 2 & -1 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

$$\det A=|A|=\begin{vmatrix} 2 & -1 & 0 & 3 \\ 1 & 1 & 5 & -2 \\ -2 & 3 & 2 & -1 \\ 1 & 0 & 4 & 0 \end{vmatrix}=-1\times\begin{vmatrix} -1 & 0 & 3 \\ 1 & 5 & -2 \\ 3 & 2 & -1 \end{vmatrix}+0-4\times\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ -2 & 3 & -1 \end{vmatrix}+0$$

$$=-1\{1-39\}-4\{10-5+15\}=38-80=-42.$$

1.8.1. Different kinds of Matrix

Row Matrix- A matrix having a single row is called Row Matrix.

$A=\begin{bmatrix} 2 & 0 & -1 & 1 \end{bmatrix}$ is a 1×4 matrix. It has one row, hence it is row matrix.

Column Matrix-A matrix having a single column is called Column Matrix.

$A=\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ is a 4×1 matrix. It has one column, hence it is column matrix.

Rectangular Matrix-If in an $m\times n$ matrix $m\neq n$ then it is called a rectangular matrix.

Square Matrix-If in an $m\times n$ matrix $m=n$ then it is called a square matrix.

$A=\begin{bmatrix} 0 & 1 & 2 & -1 \\ 4 & 0 & 3 & 1 \\ 5 & -3 & 1 & 0 \end{bmatrix}$ is a 3×4 matrix. $B=\begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & -1 \\ 0 & 5 & 1 \end{bmatrix}$ is a 3×3 matrix.

A is a rectangular matrix and B is a square matrix.

(x) $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$ (xi) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (3abc - a^3 - b^3 - c^3)$

(xii) $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$

3. Prove that $\begin{vmatrix} -a^2 & ab & ca \\ ab & -b^2 & bc \\ ca & bc & -c^2 \end{vmatrix}$ is a perfect square.

4. Prove that the determinant $\begin{vmatrix} \alpha & \sin\beta & \cos\beta \\ -\sin\beta & -\alpha & 1 \\ \cos\beta & 1 & \alpha \end{vmatrix}$ is independent of β.

5. Prove that $x = 1$ is a root of $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$.

6. Solve:

(i) $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x+\alpha & \beta & \gamma \\ \alpha & x+\beta & \gamma \\ \alpha & \beta & x+\gamma \end{vmatrix} = 0$

Ans: (i) $1, 1, -2$ (ii) $0, 0, -(\alpha+\beta+\gamma)$

7. Solve:

(i) $4x - 5y + 2z = 0,\ 2x - 7y + 4z = 0,\ x + y + z = 6.$

(ii) $2y - 3z = 0,\ x + 3y = -4,\ 3x + 4y = 3.$

(iii) $x + y - z = -3,\ 2x + 3y + z = 2,\ 8y + 3z = 1.$

(iv) $x + 2z = 7,\ 3x + y = 5,\ 2y - 3z = -5.$

(v) $x + y - z = 4,\ 2x - y + 5z = 12,\ 3x + 7y - 2z = 17.$

(vi) $12x + 9y - 7z = 2,\ 8x - 26y + 9z = 1,\ 23x + 21y - 15z = 4.$

(vii) $2x + 3y - z = 9,\ x + y + z = 9,\ 3x - y - z = -1.$

Ans: (i) $x = 1, y = 2, z = 3$; (ii) $x = 5, y = -3, z = -2$;

(iii) $x = 1, y = -1, z = 3$; (iv) $x = 1, y = 2, z = 3$;

(v) $x = 4, y = 1, z = 1$; (vi) $x = 2, y = 3, z = 7$;

(vii) $x = 2, y = 3, z = 4$.

$$\Delta_1=\begin{vmatrix}2 & 2 & -1\\ 1 & -1 & 1\\ 4 & 1 & 1\end{vmatrix}=2(-1-1)-2(1-4)+(-1)(1+4)=-3$$

$$\Delta_2=\begin{vmatrix}1 & 2 & -1\\ 1 & 1 & 1\\ 1 & 4 & 1\end{vmatrix}=1(1-4)-2(1-1)+(-1)(4-1)=-6$$

$$\Delta_3=\begin{vmatrix}1 & 2 & 2\\ 1 & -1 & 1\\ 1 & 1 & 4\end{vmatrix}=1(-4-1)-2(4-1)+2(1+1)=-7$$

Now $x=\Delta_1/\Delta=3/4$

$y=\Delta_2/\Delta=6/4$

$z=\Delta_3/\Delta=7/4$ Hence solved.

Exercise 1.10

1. Evaluate:

(i) $\begin{vmatrix}13 & 3 & 23\\ 30 & 7 & 53\\ 39 & 9 & 69\end{vmatrix}$ (ii) $\begin{vmatrix}2 & 1 & 3\\ 1 & 3 & 4\\ 3 & 2 & 1\end{vmatrix}$ (iii) $\begin{vmatrix}2 & 1 & 4\\ -3 & 0 & 2\\ 4 & 1 & -2\end{vmatrix}$

(iv) $\begin{vmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}$ (v) $\begin{vmatrix}x+1 & x+2 & x+4\\ x+3 & x+5 & x+8\\ x+7 & x+10 & x+14\end{vmatrix}$

Ans: (i) 0 (ii) –20 (iii) –14 (iv) 0 (v) –2

2. Prove by using properties:

(i) $\begin{vmatrix}1 & a & b+c\\ 1 & b & c+a\\ 1 & c & a+b\end{vmatrix}=0$ (ii) $\begin{vmatrix}b+c & c & b\\ c & c+a & a\\ b & a & a+b\end{vmatrix}=4abc$

(iii) $\begin{vmatrix}-a^2 & ab & ca\\ ab & -b^2 & bc\\ ca & bc & -c^2\end{vmatrix}=4a^2b^2c^2$

(iv) $\begin{vmatrix}a+b+2c & a & b\\ c & b+c+2a & b\\ c & a & c+a+2b\end{vmatrix}=2(a+b+c)^3$

(v) $\begin{vmatrix}a-b-c & 2b & 2c\\ 2a & b-c-a & 2c\\ 2a & 2b & c-a-b\end{vmatrix}=(a+b+c)^3$

(vi) $\begin{vmatrix}a & b-c & b+c\\ c+a & b & c-a\\ a-b & a+b & c\end{vmatrix}=(a+b+c)(a^2+b^2+c^2)$

(vii) $\begin{vmatrix}a+b+c & -c & -b\\ -c & a+b+c & -a\\ -b & -a & a+b+c\end{vmatrix}=2(a+b)(b+c)(c+a)$

(viii) $\begin{vmatrix}1 & a & a^2\\ a^2 & 1 & a\\ a & a^2 & 1\end{vmatrix}=(a^3-1)^2$ (ix) $\begin{vmatrix}\frac{1}{a} & a^2 & bc\\ \frac{1}{b} & b^2 & ca\\ \frac{1}{c} & c^2 & ab\end{vmatrix}=0$

$$\Delta_2 = \begin{vmatrix} 3 & 0 & -4 \\ 5 & 1 & 3 \\ 1 & 5 & -4 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} 3 & 1 & 0 \\ 5 & 1 & 1 \\ 1 & -3 & 5 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 3 & 1 & -4 \\ 5 & 1 & 3 \\ 1 & -3 & -4 \end{vmatrix} = 3(-4+9) - 1(-20-3) + (-4)(-15-1) = 102$$

$$\Delta_1 = \begin{vmatrix} 0 & 1 & -4 \\ 1 & 1 & 3 \\ 5 & -3 & -4 \end{vmatrix} = 0 - 1(-4-15) + (-4)(-3-5) = 51$$

$$\Delta_2 = \begin{vmatrix} 3 & 0 & -4 \\ 5 & 1 & 3 \\ 1 & 5 & -4 \end{vmatrix} = 3(-4-15) - 0 + (-4)(25-1) = -153$$

$$\Delta_3 = \begin{vmatrix} 3 & 1 & 0 \\ 5 & 1 & 1 \\ 1 & -3 & 5 \end{vmatrix} = 3(5+3) - 1(25-1) + 0 = 0$$

Now $x = \Delta_1/\Delta = 51/102 = 1/2$

$y = \Delta_2/\Delta = -153/102 = -3/2$

$z = \Delta_3/\Delta = 0/102 = 0$ Hence solved.

(vii) Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ 4 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix}$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & -1 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ 1 & 2 & 2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = 1(1-6) - 1(-2-3) + 1(4+1) = 5$$

$$\Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ 4 & -1 & 3 \\ 2 & 2 & -1 \end{vmatrix} = 3(1-6) - 1(-4-6) + 1(8+2) = 5$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & -1 \end{vmatrix} = 1(-4-6) - 3(-2-3) + 1(4-4) = 5$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ 1 & 2 & 2 \end{vmatrix} = 1(-2-8) - 1(4-4) + 3(4+1) = 5$$

Now $x = \Delta_1/\Delta = 5/5 = 1$

$y = \Delta_2/\Delta = 5/5 = 1$

$z = \Delta_3/\Delta = 5/5 = 1$ Hence solved.

(viii) Here $\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix}$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(-1-1) - 2(1-1) + (-1)(1+1) = -4$$

Exercise 1.8

1. $A=\begin{bmatrix}1 & 2 & -3\\ 2 & -5 & 3\\ 5 & -3 & -2\end{bmatrix}$ $B=\begin{bmatrix}-3 & 3 & 2\\ -2 & 5 & 3\\ 0 & 3 & -1\end{bmatrix}$ $C=\begin{bmatrix}-3 & 2 & 4\\ -5 & -2 & 1\\ 2 & -6 & -2\end{bmatrix}$

Find i) $A+B+C$ ii) $A-2B$ iii) $B+C$ iv) $A+3B-C$ v) $B-C$

vi) AB vii) BA viii) ABC ix) CAB x) AC

2. $A=\begin{bmatrix}2 & -1 & 3\\ 5 & -2 & 4\\ 1 & 0 & -1\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix}2\\ 1\\ 7\end{bmatrix}$

Write the system of equation obtained from $AX=B$

3. $A=\begin{bmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{bmatrix}$ $B=\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$

Prove that $A\times B$ is a unit matrix. Compute $B\times A, A^2, B^2$.

*****

2 TRIGONOMETRY

2.1 Revision

We have

1. $\sin^2\theta+\cos^2\theta=1$
2. $1+\cot^2\theta=\operatorname{cosec}^2\theta$
3. $\tan^2\theta+1=\sec^2\theta$

Note: $\sin\theta\times\sin\theta=(\sin\theta)^2=\sin^2\theta$; but $\sin^2\theta\neq\sin\theta^2$;

Let us express all trigonometrical ratios in **cosine ratios.**

We have $\sin^2\theta+\cos^2\theta=1$ or $\sin^2\theta=1-\cos^2\theta$

$$\sin\theta=\sqrt{1-\cos^2\theta},\ \tan\theta=\frac{\sin\theta}{\cos\theta}\ \text{or, } \tan\theta=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$

$$\cot\theta=\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\quad \operatorname{cosec}\theta=\frac{1}{\sqrt{1-\cos^2\theta}}\ \text{and}\ \sec\theta=\frac{1}{\cos\theta}$$

\# If $\sin\theta=\sin\alpha$, or $\operatorname{cosec}\theta=\operatorname{cosec}\alpha$, α being of smaller magnitude then

$$\theta=n\pi+(-1)^n\alpha$$

\# If $\cos\theta=\cos\alpha$, or $\sec\theta=\sec\alpha$, α being of smaller magnitude then

$$\theta=2n\pi\pm\alpha$$

\# If $\tan\theta=\tan\alpha$, or $\cot\theta=\cot\alpha$, α being of smaller magnitude then

$$\theta=n\pi+\alpha$$

Worked out examples :

Ex1. Prove that: i) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\sec\theta-\tan\theta$

(ii)**[QP 2009]** $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=2\operatorname{Sec}\theta$

(iii)**[QP 2013]** $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}+\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=2\operatorname{cosec}\theta$

(iv) $\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$

Find (i) $A \times B$ (ii) $B \times C$ (iii) $A \times B \times C$

Solution: (i) $A \times B = \begin{bmatrix} 6 & 4 & 20 & -6 \\ 6 & 3 & 19 & -14 \end{bmatrix}$

(ii) $B \times C = \begin{bmatrix} -5 \\ 2 \\ -1 \end{bmatrix}$

(iii) $A \times B \times C = \begin{bmatrix} 6 & 4 & 20 & -6 \\ 6 & 3 & 19 & -14 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -16 \end{bmatrix}$

Ex 6. Show that $X = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$ satisfies $X^2 - 3X + 2I = 0$, where I is the 2×2 identity matrix.

Solution: $X^2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ -6 & -2 \end{bmatrix}, 3X = \begin{bmatrix} 9 & 3 \\ -6 & 0 \end{bmatrix}, 2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Now $X^2 - 3X + 2I = = \begin{bmatrix} 7-9+2 & 3-3+0 \\ -6+6+0 & -2+0+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ex 7. Find x, y, z and t for which i) $\begin{bmatrix} x+3 & 2y+1 \\ z-3 & 2t-2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

ii) $\begin{bmatrix} x+y & y-z \\ 5-t & 7+x \end{bmatrix} = \begin{bmatrix} t-x & z-t \\ z-y & x+y+t \end{bmatrix}$

Solution: Solution: i) We make corresponding entries equal.

$x+3=2, 2y+1=1, z-3=-3, 2t-2=2$

Thus $x=-1, y=0, z=0$ and $t=2$.

ii) We make corresponding entries equal.

$x+y=t-x, y-z=z-t, 5-t=z-y$ and $7+x=x+y+t$.

$\Rightarrow 2x+y-t=0, y-2z+t=0, y-z-t+5=0, y+t-7=0.$

From 2nd and 4th equation $z=7/2$. From 3rd equation $y-t=-3/2$.
From 1st equation $x=3/4$. And we have $y+t=7, y-t=-3/2$.
$y=11/4, t=17/4$.
Hence solution is $x=3/4, y=11/4, z=7/2$ and $t=17/4$.

Ex 8. If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ prove that $A^3 = 4A$.

$A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2A.$

Hence $A^3 = A^2 \times A = 2A \times A = 2A^2 = 2 \times 2A = 4A$.

Ex 9. Find det A for $A = \begin{bmatrix} -1 & 4 & -2 \\ -2 & 10 & 2 \\ 0 & 3 & 0 \end{bmatrix}$

Solution: $\det A = \begin{vmatrix} -1 & 4 & -2 \\ -2 & 10 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 0 + (-1)^{3+2} \times 3 \times \begin{vmatrix} -1 & -2 \\ -2 & 2 \end{vmatrix} + 0 = 18$

Ex 10. Prove that $\begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & 4 \\ 6 & 3 & 4 \end{bmatrix}$ is singular.

Solution: Here $\begin{vmatrix} -1 & 1 & 2 \\ -2 & 2 & 4 \\ 6 & 3 & 4 \end{vmatrix} = 0$ since $\begin{vmatrix} -1 & 1 & 2 \\ -2 & 2 & 4 \\ 6 & 3 & 4 \end{vmatrix} = 2 \times \begin{vmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ 6 & 3 & 4 \end{vmatrix} = 2 \times 0 = 0$

(Value of a determinant having two equal rows is 0.)

$$= \begin{bmatrix} -8 & 6 & -7 & -1 \\ -8 & -2 & 3 & 1 \end{bmatrix}$$

$B \times A$ is not defined. $B \times C = \begin{bmatrix} -8 \\ 13 \\ 9 \end{bmatrix}$ $C \times B$ is not defined.

Worked out example-

Ex 1. $A = \begin{bmatrix} -1 & 2 & 1 \\ -4 & 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 & -1 \\ -4 & -6 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & -1 \end{bmatrix}$

Find (i) $A+B$ (ii) $A+3B$ (iii) $A+B-C$ (iv) $A-C$ (v) $B-2C$

Solution: (i) $A+B = \begin{bmatrix} -1 & 4 & 0 \\ -8 & -3 & 5 \end{bmatrix}$

(ii) $3B = \begin{bmatrix} 0 & 6 & -3 \\ -12 & -18 & 9 \end{bmatrix}$ $A+3B = \begin{bmatrix} -1 & 8 & -2 \\ -16 & -15 & 11 \end{bmatrix}$

(iii) $A+B-C = \begin{bmatrix} -3 & 7 & -5 \\ -7 & -5 & 6 \end{bmatrix}$

(iv) $A-C = \begin{bmatrix} -3 & 5 & -4 \\ -3 & 1 & 3 \end{bmatrix}$

(v) $2C = \begin{bmatrix} 4 & -6 & 10 \\ -2 & 4 & -2 \end{bmatrix}$ $B-2C = \begin{bmatrix} -4 & 8 & -11 \\ -2 & -10 & 5 \end{bmatrix}$

Ex 2. $A = \begin{bmatrix} 0 & 5 \\ 2 & -4 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -5 \\ -1 & 2 \\ 6 & -1 \end{bmatrix}$ $C = \begin{bmatrix} -2 & 0 \\ -1 & 2 \\ -9 & 0 \end{bmatrix}$

Show that $A+B+C=0$.

Solution: $A+B+C = \begin{bmatrix} 0+2-2 & 5-5+0 \\ 2-1-1 & -4+2+2 \\ 3+6-9 & 1-1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ex 3. $A = \begin{bmatrix} 10 & -2 & 3 \\ -3 & 5 & 1 \\ 2 & -3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 & -1 \\ 5 & -5 & 5 \\ 4 & -1 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 0 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

Show that $A+B-2C=0$

Solution: $A+B-2C = \begin{bmatrix} 10+4-14 & -2+6-4 & 3-1-2 \\ -3+5-2 & 5-5+0 & 1+5-6 \\ 2+4-6 & -3-1+4 & 5-1-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ex 4. $A = \begin{bmatrix} 0 & -5 & -2 \\ 1 & -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}$

Show that $A \times B \neq B \times A$

Solution: $A \times B = \begin{bmatrix} 0-15-8 & 0+0+2 \\ 2-3+4 & 1+0-1 \end{bmatrix} = \begin{bmatrix} -23 & 2 \\ 3 & 0 \end{bmatrix}$

$B \times A = \begin{bmatrix} 0+1 & -10-1 & -4+1 \\ 0+0 & -15+0 & -6+0 \\ 0-1 & -20+1 & -8-1 \end{bmatrix} = \begin{bmatrix} 1 & -11 & -3 \\ 0 & -15 & -6 \\ -1 & -19 & -9 \end{bmatrix}$

Hence $A \times B \neq B \times A$ proved.

Ex 5. $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 4 & -1 & 3 & 0 \\ 1 & -2 & -4 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$

Diagonal Matrix- If in a square matrix all of whose elements except the diagonal elements are zero, then the matrix is called diagonal matrix. Some elements of the diagonal may or may not be zero.

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -1\end{bmatrix} \text{ is a } 3\times 3 \text{ diagonal matrix.}$$

Zero Matrix- If every element of a matrix is zero, then it is called a zero matrix or null matrix.

$$A=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ is a } 3\times 4 \text{ zero matrix.}$$

Identity Matrix or Unit Matrix- A diagonal matrix whose elements in the principal diagonal are 1, then the matrix is called identity matrix and is denoted by I. For unit matrix $|A|=1$.

Singular and Non singular Matrix- If for a square matrix A, $|A|=0$, then A is called a Singular Matrix. If $|A|\neq 0$, then it is called a Non singular Matrix.

Symmetric Matrix- A square matrix A is a symmetric matrix, if $a_{ij}=a_{ji}$ for all i, j.

$$\text{Thus } A=\begin{bmatrix}1 & -4 & 5\\ -4 & 3 & -3\\ 5 & -3 & -1\end{bmatrix} \text{ is a symmetric matrix.}$$

Skew-symmetric Matrix- A square matrix A is a skew-symmetric matrix, if $a_{ij}=-a_{ji}$ for all i, j. It is to be noted that by definition $a_{ii}=-a_{ii}$, for diagonal elements. This implies $a_{ii}=0$. Thus diagonal elements of a skew-symmetric matrix are all 0.

$$\text{For example } A=\begin{bmatrix}0 & 4 & -2\\ -4 & 0 & -1\\ 2 & 1 & 0\end{bmatrix} \text{ is a skew-symmetric matrix.}$$

Addition of Matrix- Two matrices A and B can be added if they are of same order means they have same number of rows and same number of columns. The sum is defined as the matrix each of which elements is the sum of the corresponding elements of A and B. a_{ij} of A, b_{ij} of B, c_{ij} of C, $c_{ij}=a_{ij}+b_{ij}$

Illustration: $A=\begin{bmatrix}13 & 2 & 6\\ -1 & 3 & 2\end{bmatrix}$ $B=\begin{bmatrix}4 & -1 & 3\\ -5 & 2 & 8\end{bmatrix}$

$$A+B=\begin{bmatrix}17 & 1 & 9\\ -6 & 5 & 10\end{bmatrix} \qquad A-B=\begin{bmatrix}9 & 3 & 3\\ 4 & 1 & -6\end{bmatrix}$$

Scalar Multiplication of Matrix- By scalar multiplication of matrix we mean multiplication of the matrix $A_{m\times n}$ by some number k. It is obtained by multiplying every element a_{ij} by k.

Scalar multiple of $A=\begin{bmatrix}3 & -2 & 1\\ -1 & 0 & 5\end{bmatrix}$ by 7 is $7A=\begin{bmatrix}21 & -14 & 7\\ -7 & 0 & 35\end{bmatrix}$

a_{ij} of A corresponds $7a_{ij}$ of $7A$.

Multiplication of Matrix- Two matrices A and B can be multiplied to AB if number of columns of A is same as number of rows of B, and can be multiplied to BA if number of columns of B is same as number of rows of A. It is to be noted that $AB\neq BA$

For a_{ij} of $A_{m\times n}$, b_{ij} of $B_{n\times l}$, c_{ij} of $A\times B$ (or AB) $c_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$ $A\times B$ is of order $m\times l$.

Illustration: $A=\begin{bmatrix}3 & -2 & 1\\ 1 & -2 & 3\end{bmatrix}$ $B=\begin{bmatrix}1 & 2 & -2 & 1\\ 6 & -1 & 2 & 3\\ 1 & -2 & 3 & 2\end{bmatrix}$ $C=\begin{bmatrix}2\\ -1\\ 3\\ -2\end{bmatrix}$

$A\times B$

$$=\begin{bmatrix}3.1+(-2)6+1.1 & 3.2+(-2)(-1)+1(-2) & 3(-2)+(-2)2+1.3 & 3.1+(-2)3+1.2\\ 1.1+(-2)6+3.1 & 1.2+(-2)(-1)+3(-2) & 1(-2)+(-2)2+3.3 & 1.1+(-2)3+3.2\end{bmatrix}$$

Ex 9. Find the value:

(i) **[QP 2013]** $\tan(-960^\circ)$ (ii) **[QP 2009]** $\cos(-225^\circ)$
(iii) **[QP 2011]** $\cos(-1290^\circ)$ (iv) **[QP 2011]** $\tan(-1485^\circ)$
(v) **[QP 2010]** $\sin(-660^\circ)$ (vi) **[QP 2013]** $\text{cosec}(-660^\circ)$
(vii) **[QP 2013]** $\cot(-1575^\circ)$ (viii) **[QP 2014]** $\tan(-1125^\circ)$
(ix) **[QP 2014]** $\sin(570^\circ)$ (x) **[QP 2015]** $\text{cosec}\ 1470^\circ$
(xi) **[QP 2015]** $\tan 810^\circ$ (xii) **[QP 2015]** $\sin(-120^\circ)$

Soln: (i) $\tan(-960^\circ) = -\tan(960^\circ) = -\tan(3\times360^\circ-120^\circ) = \tan(120^\circ)$

$= \tan(180^\circ-60^\circ) = -\tan 60^\circ = -\sqrt{3}$

(ii) $\cos(-225^\circ) = \cos(225^\circ) = \cos(180^\circ+45^\circ) = -\cos 45^\circ = -\dfrac{1}{\sqrt{2}}$

(iii) $\cos(-1290^\circ) = \cos(1290^\circ) = \cos(3\times360^\circ+210^\circ) = \cos(210^\circ)$

$\cos(180^\circ+30^\circ) = -\cos 30^\circ = -\dfrac{\sqrt{3}}{2}$

(iv) $\tan(-1485^\circ) = -\tan 1485^\circ = -\tan(4\times360^\circ+45^\circ) = -\tan 45^\circ = -1$

(v) $\sin(-660^\circ) = -\sin(660^\circ) = -\sin(2\times360^\circ-60^\circ) = \sin 60^\circ = \dfrac{\sqrt{3}}{2}$

(vi) $\text{cosec}(-660^\circ) = -\text{cosec}(660^\circ) = -\text{cosec}(2\times360^\circ-60^\circ) = \text{cosec}\ 60^\circ$

$= \dfrac{2}{\sqrt{3}}$

(vii) $\cot(-1575^\circ) = -\cot(1575^\circ) = -\cot(4\times360^\circ+135^\circ) = -\cot 135^\circ$

$= -\cot(180^\circ-45^\circ) = \cot 45^\circ = 1$

(viii) $\tan(-1125^\circ) = -\tan 1125^\circ = -\tan(3\times360^\circ+45^\circ) = -\tan 45^\circ = -1$

(ix) $\sin 570^\circ = \sin(360^\circ+210^\circ) = \sin 210^\circ = \sin(180^\circ+30^\circ)$

$= -\sin 30^\circ = -\dfrac{1}{2}$

(x) $\text{cosec}\ 1470^\circ = \text{cosec}(4\times360^\circ+30^\circ) = \text{cosec}\ 30^\circ = 2$

(xi) $\tan 810^\circ = \tan(2\times360^\circ+90^\circ) = \tan 90^\circ = \alpha$

(xii) $\sin(-120^\circ) = -\sin 120^\circ = -\sin 120^\circ = -\sin(180^\circ-60^\circ)$

$= -\sin 60^\circ = -\dfrac{\sqrt{3}}{2}$

Ex 10. prove that $\sin^2\dfrac{\pi}{4} + \sin^2\dfrac{3\pi}{4} + \sin^2\dfrac{5\pi}{4} + \sin^2\dfrac{7\pi}{4} = 2$

Soln: LHS $= \sin^2\dfrac{\pi}{4} + \sin^2\dfrac{3\pi}{4} + \sin^2\dfrac{5\pi}{4} + \sin^2\dfrac{7\pi}{4}$

$= \sin^2\dfrac{\pi}{4} + \sin^2(\pi-\dfrac{\pi}{4}) + \sin^2(\pi+\dfrac{\pi}{4}) + \sin^2(2\pi-\dfrac{\pi}{4})$

$= \sin^2\dfrac{\pi}{4} + \sin^2\dfrac{\pi}{4} + (-1)^2\sin^2\dfrac{\pi}{4} + (-1)^2\sin^2\dfrac{\pi}{4}$

$= \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} = 2 = \text{RHS}$

Ex 11. Prove that: (i) $\sin(90^\circ+\theta)\sin(270^\circ+\theta) + \sin(270^\circ+\theta)\cos(180^\circ+\theta) = 0$

(ii) **[QP 2010]** $\sin^2 48^\circ + \sin^2 42^\circ = 1$
(iii) **[QP 2009, 2013]** $\sin^2 36^\circ + \sin^2 54^\circ = 1$
(iv) **[QP 2010]** $\sin^2 18^\circ + \sin^2 72^\circ = 1$
(v) **[QP 2009]** $\sin^2 49^\circ + \sin^2 41^\circ = 1$

Soln: LHS $= \sin(90^\circ+\theta)\sin(270^\circ+\theta) + \sin(270^\circ+\theta)\cos(180^\circ+\theta)$

$= \cos\theta(-\cos\theta) + (-\cos\theta)(-\cos\theta)$

$= -\cos^2\theta + \cos^2\theta = 0 = \text{RHS}$ Proved.

(ii) LHS $= \sin^2 48^\circ + \sin^2 42^\circ = \sin^2(90^\circ-48^\circ) + \sin^2 42^\circ$

$= \cos^2 42^\circ + \sin^2 42^\circ = 1 = \text{RHS}$ Proved.

(iii) LHS $= \sin^2 36^\circ + \sin^2 54^\circ = \sin^2(90^\circ-54^\circ) + \sin^2 54^\circ$

$= \cos^2 54^\circ + \sin^2 54^\circ = 1 = \text{RHS}$ Proved.

(iv) LHS $= \sin^2 18^\circ + \sin^2 72^\circ = \sin^2(90^\circ-72^\circ) + \sin^2 72^\circ$

$= \cos^2 72^\circ + \sin^2 72^\circ = 1 = \text{RHS}$ Proved.

(v) LHS $= \sin^2 49^\circ + \sin^2 41^\circ = \sin^2(90^\circ-41^\circ) + \sin^2 41^\circ$

$= \cos^2 41^\circ + \sin^2 41^\circ = 1 = \text{RHS}$ Proved.

Ex 12. [QP 2010] Prove that $\dfrac{\sin 250^\circ + \tan 290^\circ}{\cot 200^\circ + \cos 340^\circ} = -1$

Soln: $\sin 250^\circ = \sin(270^\circ-20^\circ) = -\cos 20^\circ$

$\tan 290^\circ = \tan(270^\circ+20^\circ) = -\cot 20^\circ$

$\cos 340^\circ = \cos(360^\circ-20^\circ) = \cos 20^\circ$

$\cot 200^\circ = \cot(180^\circ+20^\circ) = \cot 20^\circ$

Now LHS $= \dfrac{\sin 250^\circ + \tan 290^\circ}{\cot 200^\circ + \cos 340^\circ} = \dfrac{-\cos 20^\circ - \cot 20^\circ}{\cot 20^\circ + \cos 20^\circ}$

Soln: (i) We have $\cos^2A = 1-\sin^2A = 1-\frac{3}{4}=\frac{1}{4}$ (given $\sin A = \frac{\sqrt{3}}{2}$)

$\Rightarrow \cos A = \frac{1}{2}$ $\therefore \sec A = 2$ and $\tan A = \frac{\sin A}{\cos A} = \sqrt{3}$

(ii) Given $\cos\theta = -\frac{4}{5}$ We have $\sin^2\theta = 1-\cos^2\theta = 1-(-\frac{4}{5})^2$

$\Rightarrow \sin\theta = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$ Now, $\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{4}{3}$

(iii) Given $\tan\theta = \frac{5}{4}$ or, $\cot\theta = \frac{4}{5}$

We know $\tan^2\theta + 1 = \sec^2\theta$ or, $\left(\frac{5}{4}\right)^2 + 1 = \sec^2\theta \Rightarrow \sec^2\theta = \frac{41}{16}$

$\Rightarrow \cos^2\theta = \frac{16}{41} \Rightarrow \cos\theta = \frac{4}{\sqrt{41}}$

Again, $\sin^2\theta = 1-\cos^2\theta = 1-\frac{16}{41} = \frac{25}{41} \Rightarrow \sin\theta = \frac{5}{\sqrt{41}}$

(iv) We know, $\cos^2A = 1-\sin^2A = 1-(\frac{3}{5})^2 = 1-\frac{9}{25} = \frac{16}{25}$ (given $\sin A = \frac{3}{5}$)

$\Rightarrow \cos A = \frac{4}{5}$ $\therefore \tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$

Also $\sin^2B = 1-\cos^2B = 1-(\frac{12}{13})^2 = 1-\frac{144}{169} = \frac{25}{169}$ (given $\cos B = \frac{12}{13}$)

$\sin B = \frac{5}{13}$ $\therefore \tan B = \frac{\sin B}{\cos B} = \frac{5}{12}$

Now $\frac{\tan A - \tan B}{1+\tan A\tan B} = \frac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}.\frac{5}{12}} = \frac{\frac{9-5}{12}}{\frac{48+15}{48}} = \frac{16}{63}$

$\therefore \frac{\tan A - \tan B}{1+\tan A\tan B} = \frac{16}{63}$

(v) Given $\sin\theta = \frac{4}{5} \Rightarrow \text{cosec}\theta = \frac{5}{4}$

We have $\cos^2\theta = 1-\sin^2\theta = 1-(\frac{4}{5})^2 = \frac{9}{25}$

$\Rightarrow \cos\theta = \frac{3}{5}$ $\Rightarrow \text{Sec}\theta = \frac{5}{3}$

(vi) Given $\cos\theta = \frac{3}{5}$

We have $\sin^2\theta = 1-\cos^2\theta = 1-(\frac{3}{5})^2 = 1-\frac{9}{25} = \frac{16}{25}$

$\therefore \sin\theta = \frac{4}{5}$ $\Rightarrow \text{cosec}\theta = \frac{5}{4}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4}{3}$

Ex 7. If $\sin^4\theta + \sin^2\theta = 1$ then prove that $\tan^4\theta - \tan^2\theta = 1$

Soln: $\sin^4\theta + \sin^2\theta = 1$

$\Rightarrow (1-\cos^2\theta)^2 + 1 - \cos^2\theta = 1$

$\Rightarrow (1-\frac{1}{\sec^2\theta})^2 + 1 - \frac{1}{\sec^2\theta} = 1$

$\Rightarrow \left(\frac{\sec^2\theta - 1}{\sec^2\theta}\right)^2 + 1 - \frac{1}{\sec^2\theta} = 1$

$\Rightarrow \frac{\tan^4\theta}{(1+\tan^2\theta)^2} - \frac{1}{1+\tan^2\theta} = 0$

$\Rightarrow \frac{\tan^4\theta - 1 - \tan^2\theta}{(1+\tan^2\theta)^2} = 0 \Rightarrow \tan^4\theta - \tan^2\theta = 1$ Proved.

Ex 8. [QP 2012] In a triangle ABC, if $\angle C = 90°$; AC = 4 unit and BC = 3 unit find AB.

Proof: We have, $(AC)^2 + (BC)^2 = (AB)^2$

$\Rightarrow (3)^2 + (4)^2 = (AB)^2$

$\Rightarrow (5)^2 = (AB)^2$ $\Rightarrow AB = 5$ unit

Proof : Given, $\tan\theta = c$ $\therefore \cot\theta = \frac{1}{c}$,

we have $1+\cot^2\theta = \text{cosec}^2\theta$

So $1+\left(\frac{1}{c}\right)^2 = \text{cosec}^2\theta \Rightarrow \frac{1+c^2}{c^2} = \text{cosec}^2\theta$

$\therefore \frac{c^2}{c^2+1} = \sin^2\theta \Rightarrow \sin\theta = \sqrt{\frac{c^2}{c^2+1}}$ or $\sin\theta = \frac{c}{\sqrt{c^2+1}}$ Proved.

Ex 3. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$ then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

Proof: given, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

or, $\sin\theta = \sqrt{2}\cos\theta - \cos\theta$

$\Rightarrow (\sqrt{2}+1)\sin\theta = (\sqrt{2}+1)(\sqrt{2}-1)\cos\theta$

$\Rightarrow (\sqrt{2}+1)\sin\theta = (2-1)\cos\theta$

$\Rightarrow (\sqrt{2}+1)\sin\theta = \cos\theta \Rightarrow \sqrt{2}\sin\theta + \sin\theta = \cos\theta$

$\Rightarrow \sqrt{2}\sin\theta = \cos\theta - \sin\theta$ Proved.

Ex 4. Eliminate θ from the following equation:

(i) $x = r\cos\theta$ and $y = r\sin\theta$

(ii) **[QP 2009, 2013]** $x = a\tan\theta$ and $y = b\sec\theta$

Soln: (i) $x = r\cos\theta \Rightarrow x^2 = r^2\cos^2\theta$ ....(I)

$y = r\sin\theta \Rightarrow y^2 = r^2\sin^2\theta$ ....(II)

(I) + (II) $\Rightarrow x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\Rightarrow x^2+y^2 = r^2$

(ii) $x = a\tan\theta \Rightarrow x^2 = a^2\tan^2\theta$ ....(I)

$y = b\sec\theta \Rightarrow y^2 = b^2\sec^2\theta$ ....(II)

We have, $1+\tan^2\theta = \sec^2\theta \Rightarrow 1+\frac{x^2}{a^2} = \frac{y^2}{b^2}$ (from I and II)

Ex 5. [QP 2010] If $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$ then prove that $x^2+y^2+z^2 = r^2$

Soln: Given $x = r\cos\theta\cos\phi \Rightarrow \frac{x}{r} = \cos\theta\cos\phi$

$\therefore (\frac{x}{r})^2 = (\cos\theta\cos\phi)^2 \Rightarrow (\frac{x}{r})^2 = \cos^2\theta\cos^2\phi$ ....(I)

Again given $y = r\cos\theta\sin\phi \Rightarrow \frac{y}{r} = \cos\theta\sin\phi$

$\therefore (\frac{y}{r})^2 = (\cos\theta\sin\phi)^2 \Rightarrow (\frac{y}{r})^2 = \cos^2\theta\sin^2\phi$ ....(II)

(I)+(II) $\Rightarrow (\frac{x}{r})^2 + (\frac{y}{r})^2 = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi$

$= \cos^2\theta(\cos^2\phi + \sin^2\phi) = \cos^2\theta$

$\therefore (\frac{x}{r})^2 + (\frac{y}{r})^2 = \cos^2\theta$ ....(III)

Again $z = r\sin\theta \Rightarrow \frac{z}{r} = \sin\theta$ $\therefore (\frac{z}{r})^2 = \sin^2\theta$ ....(IV)

(III) + (IV) $\Rightarrow (\frac{x}{r})^2 + (\frac{y}{r})^2 + (\frac{z}{r})^2 = \cos^2\theta + \sin^2\theta = 1$

$\Rightarrow x^2+y^2+z^2 = r^2$

Ex 6. Find the value of: (i) **[QP 2010]** secA and tanA if $\sin A = \frac{\sqrt{3}}{2}$

(ii) **[QP 2011]** $\cot\theta$ if $\cos\theta = -\frac{4}{5}$

(iii) **[QP 2009, 2013]** $\sin\theta$ and $\cos\theta$ if $\tan\theta = \frac{5}{4}$ or $\cot\theta = \frac{4}{5}$

(iv) $\frac{\tan A - \tan B}{1+\tan A\tan B}$ if $\sin A = \frac{3}{5}$, $\cos B = \frac{12}{13}$

(v) **[QP 2011]** $\sec\theta$ and $\text{cosec}\theta$ if $\sin\theta = \frac{4}{5}$

(vi) **[QP 2010]** $\text{cosec}\theta$ and $\tan\theta$ if $\cos\theta = \frac{3}{5}$

(v) $(\sin\theta - \cos\theta)^2 = 1 - 2\sin\theta\cos\theta$

(vi) $\cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(vii) $(1+\tan^2 A)(1-\sin^2 A) = 1$

(viii) $\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$

(ix) **[QP 2015]** $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}$

Proof: (i) LHS $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\dfrac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$

$= \sqrt{\dfrac{(1-\sin\theta)^2}{(1-\sin^2\theta)}} = \dfrac{1-\sin\theta}{\cos\theta} = \dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS}$ Proved.

(ii) L.H.S $= \sqrt{\dfrac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}} + \sqrt{\dfrac{(1-\sin\theta)^2}{(1+\sin\theta)(1-\sin\theta)}}$

$= \sqrt{\dfrac{(1+\sin\theta)^2}{(1-\sin^2\theta)}} + \sqrt{\dfrac{(1-\sin\theta)^2}{((1-\sin^2\theta)}}$

$= \dfrac{(1+\sin\theta)}{\cos\theta} + \dfrac{(1-\sin\theta)}{\cos\theta} = \dfrac{1+\sin\theta+1-\sin\theta}{\cos\theta} = \dfrac{2}{\cos\theta} = 2\sec\theta = \text{RHS}$

Proved.

(iii) LHS $= \sqrt{\dfrac{(1+\cos\theta)(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}} + \sqrt{\dfrac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}}$

$= \sqrt{\dfrac{(1+\cos\theta)^2}{(1-\cos^2\theta)}} + \sqrt{\dfrac{(1-\cos\theta)^2}{(1-\cos^2\theta)}} = \sqrt{\dfrac{(1+\cos\theta)^2}{\sin^2\theta}} + \sqrt{\dfrac{(1-\cos\theta)^2}{\sin^2\theta}}$

$= \dfrac{(1+\cos\theta)}{\sin\ \theta} + \dfrac{(1-\cos\theta)}{\sin\ \theta} = \dfrac{1+\cos\theta+1-\cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta}$

$= 2\operatorname{cosec}\theta = \text{RHS}$ Proved

(iv) L.H.S. $= \dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$

$= \dfrac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} + \dfrac{1+\cos\theta}{\sin\theta} = \dfrac{\sin\theta(1-\cos\theta)}{\sin^2\theta} + \dfrac{1+\cos\theta}{\sin\theta}$

$= \dfrac{1-\cos\theta+1+\cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta} = 2\operatorname{cosec}\theta = \text{RHS}$ Proved.

(v) LHS $= (\sin\theta - \cos\theta)^2 = \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = 1 - 2\sin\theta\cos\theta = \text{RHS}$
Proved.

(vi) RHS $= \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}} = \dfrac{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} = \cos^2\theta - \sin^2\theta = \text{LHS}$ Proved.

(vii) LHS $= (1+\tan^2 A)(1-\sin^2 A) = \sec^2 A.\cos^2 A$

$= \dfrac{1}{\cos^2 A}.\cos^2 A = 1 = \text{RHS}$ Proved.

(viii) LHS $= \sec^4\theta - \sec^2\theta = \sec^2\theta(\sec^2\theta - 1) = \sec^2\theta\ .\tan^2\theta$

$= (1+\tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta = \text{RHS}$ Proved.

(ix) $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \dfrac{(\sec\theta+\tan\theta)\{1-(\sec\theta-\tan\theta)\}}{\tan\theta-\sec\theta+1}$

$= \dfrac{(\sec\theta+\tan\theta)(\tan\theta-\sec\theta+1)}{\tan\theta-\sec\theta+1} = \sec\theta + \tan\theta$

$= \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1+\sin\theta}{\cos\theta}$

$= \dfrac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta(1-\sin\theta)} = \dfrac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}$

$= \dfrac{\cos^2\theta}{\cos\theta(1-\sin\theta)} = \dfrac{\cos\theta}{1-\sin\theta}$ Proved.

Ex 2. If $\tan\theta = c$ then prove that $\sin\theta = \dfrac{c}{\sqrt{c^2+1}}$

(ii) $\tan 37^\circ = \tan(45^\circ - 8^\circ) = \dfrac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ} = \dfrac{1 - \tan 8^\circ}{1 + \tan 8^\circ}$

$$= \frac{1 - \dfrac{\sin 8^\circ}{\cos 8^\circ}}{1 + \dfrac{\sin 8^\circ}{\cos 8^\circ}} = \frac{\dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}{\dfrac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}} = \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} \quad \text{proved.}$$

(iii) $\tan 53^\circ = \tan(45^\circ + 8^\circ) = \dfrac{\tan 45^\circ + \tan 8^\circ}{1 - \tan 45^\circ \tan 8^\circ} = \dfrac{1 + \tan 8^\circ}{1 - \tan 8^\circ}$

$$= \frac{1 + \dfrac{\sin 8^\circ}{\cos 8^\circ}}{1 - \dfrac{\sin 8^\circ}{\cos 8^\circ}} = \frac{\dfrac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}}{\dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}} = \frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ - \sin 8^\circ} \quad \text{proved.}$$

(iv) $\tan 36^\circ = \tan(45^\circ - 9^\circ) = \dfrac{\tan 45^\circ - \tan 9^\circ}{1 + \tan 45^\circ \tan 9^\circ} = \dfrac{1 - \tan 9^\circ}{1 + \tan 9^\circ}$

$$= \frac{1 - \dfrac{\sin 9^\circ}{\cos 9^\circ}}{1 + \dfrac{\sin 9^\circ}{\cos 9^\circ}} = \frac{\dfrac{\cos 9^\circ - \sin 9^\circ}{\cos 9^\circ}}{\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ}} = \frac{\cos 9^\circ - \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} \quad \text{proved.}$$

Ex 4. Prove: **(i)** $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ **(ii)** $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(iii) $\operatorname{cosec}(A+B) = \dfrac{\operatorname{cosec} A \operatorname{cosec} B}{\cot A + \cot B}$ **(iv)** $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(v) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(vi) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(vii) $\tan(A+B)\tan(A-B) = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

(viii) $\sin(45^\circ + \theta)\sin(45^\circ - \theta) = \dfrac{1}{2} - \sin^2\theta$

(ix) $\tan(45^\circ + \theta)\tan(45^\circ - \theta) = 1$

(x)[QP 2014] $\dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} + \dfrac{\sin(A-B)}{\cos A \cos B} = 0$

Proof : (i) LHS $= \tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$$= \frac{\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \text{RHS}$$

(ii) LHS $= \tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)} = \dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B + \sin A \sin B}{\cos A \cos B}} = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \text{RHS}$$

(iii) LHS $= \operatorname{cosec}(A+B) = \dfrac{1}{\sin(A+B)} = \dfrac{1}{\sin A \cos B + \cos A \sin B}$

$$= \frac{\dfrac{1}{\sin A \sin B}}{\dfrac{\sin A \cos B + \cos A \sin B}{\sin A \sin B}} = \frac{\operatorname{cosec} A \operatorname{cosec} B}{\cot B + \cot A} = \text{RHS}$$

(iv) LHS $= \cot(A-B) = \dfrac{\cos(A-B)}{\sin(A-B)} = \dfrac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$

$$= \frac{\dfrac{\cos A \cos B + \sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B - \cos A \sin B}{\sin A \sin B}} = \frac{\cot A \cot B + 1}{\cot B - \cot A} = \text{RHS}$$

(v) $\sin(A+B)\sin(A-B) = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$
$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)$

2.2 Compound angles

2.2.1 The angles which are formed by addition and subtraction of two or more angles are known as compound angles .

e.g. For the angles A , B , C , the angles A+B , A–B , B+C , B–C are compound angles

2.2.2 Addition and Subtraction formulae for compound angles :

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Cor :

(i) $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

(ii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$

Worked out examples :

Ex 1.(i)[Q.P 2011] Find the value of sin 15°

(ii)[Q.P 2013] Find the value of tan 15°

(iii)[Q.P 2010] Find the value of tan75°

(iv) [Q.P 2009] Evaluate sin75° and cos15°

Soln: (i) $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\tan 15^\circ = \tan(45^\circ - 30^\circ) = \dfrac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \dfrac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

(iii) $\tan 75^\circ = \tan(30^\circ + 45^\circ) = \dfrac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ} = \dfrac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(iv) $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Ex 2.[QP 2013] Evaluate cos 15° and hence find sin105°

Soln: $\sin 105^\circ = \sin(90^\circ + 15^\circ) = \cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Ex 3.(i)[QP 2011] Prove that $\tan(45^\circ - \theta) = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

(ii))[QP 2010] Prove that $\tan 37^\circ = \dfrac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ}$

(iii)[QP 2014] Prove that $\tan 53^\circ = \dfrac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ - \sin 8^\circ}$

(iv)[QP 2015] Prove that $\tan 36^\circ = \dfrac{\cos 9^\circ - \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ}$

Proof : (i) $\tan(45^\circ - \theta) = \dfrac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta} = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

$$= \frac{1 - \frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin\theta}{\cos\theta}} = \frac{\frac{\cos\theta - \sin\theta}{\cos\theta}}{\frac{\cos\theta + \sin\theta}{\cos\theta}} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \quad \text{proved.}$$

Exercise 2.1

1. Prove that $(\cot\theta + \operatorname{cosec}\theta)^2 = \dfrac{1+\cos\theta}{1-\cos\theta}$
2. Prove that $(\sec\theta + \tan\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$
3. Prove that $\tan\theta + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta$
4. Prove that $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} - \sec\theta = \sec\theta - \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$
5. Prove that $\cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
6. Prove that $\cos^4\theta + \sin^4\theta = 1 - 3\cos^2\theta\sin^2\theta$
7. If $\sin^2\theta + \cos^2\theta = 1$ then prove that $1+\cot^2\theta = \operatorname{cosec}^2\theta$
8. If $\sin^2\theta + \cos^2\theta = 1$ then prove that $\tan^2\theta + 1 = \sec^2\theta$
9. If $\cos^2\theta + \cos^4\theta = 1$ then prove that
   (i) $\tan^2\theta + \tan^4\theta = 1$ (ii) $\sin\theta + \sin^2\theta = 1$
10. Prove that $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$
11. If $\sin\theta + \sin^2\theta = 1$ then prove that $\cos^2\theta + \cos^4\theta = 1$
12. If $7\sin^2\theta + 3\cos^2\theta = 4$ then prove that $\tan\theta = \pm\dfrac{1}{\sqrt{3}}$
13. If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$ then prove that $m^2 - n^2 = 4\sqrt{mn}$
14. Prove that $\sec\theta\operatorname{cosec}\theta = \tan^2\theta\cot\theta + \cot^2\theta\tan\theta$.
15. If $\sin\theta = \dfrac{3}{5}$ then find other trigonometrical ratio of θ.

    Ans. $\cos\theta = \dfrac{4}{5}$; $\tan\theta = \dfrac{3}{4}$; $\operatorname{cosec}\theta = \dfrac{5}{3}$; $\sec\theta = \dfrac{5}{4}$; $\cot\theta = \dfrac{4}{3}$
16. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$ then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$
17. If $\operatorname{cosec}\theta + \sec\theta = n$ and $\cos\theta + \sin\theta = m$ then prove that $n(m^2-1) = 2m$
18. If $\cos\varphi = \dfrac{\cos\alpha + \cos\beta}{1+\cos\alpha\cos\beta}$ then prove that $\sin\varphi = \pm\dfrac{\sin\alpha\sin\beta}{1+\cos\alpha\cos\beta}$

19. Express $(1 \pm 2\sin\alpha\cos\alpha)$ as a perfect square.
20. Prove that $\sec^4\theta + \tan^4\theta = 1 + 2\sec^2\theta\tan^2\theta$
21. If $3\sin\theta + 5\cos\theta = 5$ then prove that $3\cos\theta - 5\sin\theta = \pm 3$
22. Prove that $\dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\operatorname{cosec}^2\theta} = 1$
23. Prove that $\dfrac{\cos\theta + \cos\phi}{\sin\theta - \sin\phi} = \dfrac{\sin\theta + \sin\phi}{\cos\phi - \cos\theta}$
24. Prove that $\dfrac{\cos x}{\sin x + \cos y} + \dfrac{\cos y}{\sin y - \cos x} = \dfrac{\cos x}{\sin x - \cos y} + \dfrac{\cos y}{\sin y + \cos x}$
25. Solve: (i) $2\cos^2\theta - 3\cos\theta + 1 = 0$ Ans: $\theta = 2n\pi$, $\theta = 2n\pi \pm 60^\circ$
    (ii) $\sec^2\theta = 2$ Ans: $n\pi \pm 45^\circ$
    (iii) $\cot^2\theta + \operatorname{cosec}^2\theta = 3$ Ans: $n\pi \pm 45^\circ$
    (iv) $2\cos^2\theta + 4\sin^2\theta = 3$ Ans: $n\pi \pm 45^\circ$
    (v) $4\cos^2\theta + 6\sin^2\theta = 5$ Ans: $n\pi \pm 45^\circ$

* * *

$= \frac{-(\cos 20^\circ + \cot 20^\circ)}{\cot 20^\circ + \cos 20^\circ} = -1 = RHS$ Proved.

Ex 13. [QP 2013] Solve: $2\sin^2\theta + 3\cos\theta = 0$

Soln: $2\sin^2\theta + 3\cos\theta = 0$

$\Rightarrow 2(1-\cos^2\theta) + 3\cos\theta = 0$

$\Rightarrow 2\cos^2\theta - 3\cos\theta - 2 = 0$

$\Rightarrow 2\cos^2\theta - 4\cos\theta + \cos\theta - 2 = 0$

$\Rightarrow (\cos\theta - 2)(2\cos\theta + 1) = 0$

$\cos\theta = 2$ is impossible.

$\theta = -\frac{1}{2} = \cos(\pi - \frac{\pi}{3}) = \cos\frac{2\pi}{3}$. Thus $\theta = 2n\pi \pm \frac{2\pi}{3}$

Ex 14. [QP 2014] Solve: $2\cos^2\theta + \sin\theta - 1 = 0$

Soln: $2\cos^2\theta + \sin\theta - 1 = 0$

$\Rightarrow 2(1-\sin^2\theta) + \sin\theta - 1 = 0$

$\Rightarrow 2\sin^2\theta - \sin\theta - 1 = 0$

$\Rightarrow 2\sin^2\theta - 2\sin\theta + \sin\theta - 1 = 0$

$\Rightarrow (\sin\theta - 1)(2\sin\theta + 1) = 0$

$\sin\theta = 1 = \sin\frac{\pi}{2} \Rightarrow \theta = 2n\pi + \frac{\pi}{2}$

$\sin\theta = -\frac{1}{2} = \sin(-\frac{\pi}{6})$. Thus $\theta = n\pi + (-1)^n(-\frac{\pi}{6})$

Ex 15. Solve: (i) $2\sin^2\theta + \sin\theta = 3$ $\quad -100^\circ < \theta < 1000^\circ$

(ii) $\tan ax = \cot bx$

(iii) $3(\sec^2\theta + \tan^2\theta) = 5$

(iv) $\cot\theta + \tan\theta = 2\sec\theta$ $\quad \pi < \theta < 6\pi$

(v) $\cos^2\theta - \sin\theta - 1 = \frac{1}{4}$

Soln: (i) $2\sin^2\theta + \sin\theta = 3$

$\Rightarrow 2\sin^2\theta + \sin\theta - 3 = 0$

$\Rightarrow 2\sin^2\theta - 2\sin\theta + 3\sin\theta - 3 = 0$

$\Rightarrow (\sin\theta - 1)(2\sin\theta + 3) = 0$

$\sin\theta = 1 = \sin\frac{\pi}{2} \Rightarrow \theta = 2\pi + \frac{\pi}{2} = 450^\circ, 4\pi + \frac{\pi}{2} = 810^\circ$

$2\sin\theta + 3 = 0$ is not possible.

(ii) $\tan ax = \tan(\frac{\pi}{2} - bx)$

$\Rightarrow ax = n\pi + (\frac{\pi}{2} - bx) \Rightarrow x = \frac{2n+1}{a+b}\cdot\frac{\pi}{2}$

(iii) $3(\sec^2\theta + \tan^2\theta) = 5$

$\Rightarrow 3(1 + 2\tan^2\theta) = 5$

$\Rightarrow 6\tan^2\theta = 2$

$\Rightarrow \tan\theta = \pm\frac{1}{\sqrt{3}} = \tan(\pm\frac{\pi}{6})$

$\Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

(iv) $\cot\theta + \tan\theta = 2\sec\theta$ $\quad \pi < \theta < 6\pi$

$\Rightarrow \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{2}{\cos\theta}$

$\Rightarrow \frac{1}{\sin\theta\cos\theta} = \frac{2}{\cos\theta}$

$\Rightarrow \sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$ $\quad \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}$

Since $\pi < \theta < 6\pi$ So $\theta = 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, 5\pi - \frac{\pi}{6}$.

(v) $\cos^2\theta - \sin\theta - 1 = \frac{1}{4}$

$\Rightarrow 1 - \sin^2\theta - \sin\theta - 1 = \frac{1}{4}$

$\Rightarrow 4\sin^2\theta + 4\sin\theta + 1 = 0$

$\Rightarrow (2\sin\theta + 1)^2 = 0 \Rightarrow \sin\theta = -\frac{1}{2} = \sin(-\frac{\pi}{6})$

$\Rightarrow \theta = n\pi + (-1)^n(-\frac{\pi}{6})$

$$= 2\cos\frac{65^\circ + 25^\circ}{2}\cos\frac{65^\circ - 25^\circ}{2} = 2\cos\frac{90^\circ}{2}\cos\frac{40^\circ}{2}$$

$$= 2\cos 45^\circ \cos 20^\circ = 2\frac{1}{\sqrt{2}}\cos 20^\circ = \sqrt{2}\cos 20^\circ = \text{RHS}$$

(ii) $\text{LHS} = \cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 2\cos\frac{120^0}{2}\cos\frac{80^0 - 40^0}{2} - \cos 20^\circ$

$$= 2\cos 60^\circ \cos 20^\circ - \cos 20^\circ = 2\times\frac{1}{2}\cos 20^\circ - \cos 20^\circ$$

$$= \cos 20^\circ - \cos 20^\circ = 0 = \text{RHS}$$

(iii) $\text{LHS} = \cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 2\cos\frac{120^0}{2}\cos\frac{20^0 - 100^0}{2} + \cos 140^\circ$

$$= 2\cos 60^\circ \cos(-40^\circ) + \cos(180^\circ - 40^\circ) = 2\times\frac{1}{2}\cos 40^\circ - \cos 140^\circ$$

$$= \cos 40^\circ - \cos 40^\circ = 0 = \text{RHS}$$

(iv) $\text{LHS} = \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$$= \frac{1}{2}\cos 20^\circ \cos 40^\circ \cos 80^\circ$$

$$= \frac{1}{4}(2\cos 20^\circ \cos 40^\circ)\cos 80^\circ$$

$$= \frac{1}{4}(\cos 60^\circ + \cos 20^\circ)\cos 80^\circ$$

$$= \frac{1}{4}\cos 60^\circ \cos 80^\circ + \frac{1}{4}\cos 20^\circ \cos 80^\circ$$

$$= \frac{1}{4}\times\frac{1}{2}\cos 80^\circ + \frac{1}{4}\times\frac{1}{2}(2\cos 20^\circ \cos 80^\circ)$$

$$= \frac{1}{8}\cos(180^\circ - 100^\circ) + \frac{1}{8}(\cos 100^\circ + \cos 60^\circ)$$

$$= -\frac{1}{8}\cos 100^\circ + \frac{1}{8}\cos 100^\circ + \frac{1}{8}\times\frac{1}{2} = \frac{1}{16} = \text{RHS}$$

(v) $\text{LHS} = \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$$= \frac{\sqrt{3}}{2}\sin 20^\circ \sin 40^\circ \sin 80^\circ$$

$$= \frac{\sqrt{3}}{4}(2\sin 20^\circ \sin 40^\circ)\sin 80^\circ$$

$$= \frac{\sqrt{3}}{4}(\cos 20^\circ - \cos 60^\circ)\sin 80^\circ$$

$$= \frac{\sqrt{3}}{4}\cos 20^\circ \sin 80^\circ - \frac{\sqrt{3}}{4}\cos 60^\circ \sin 80^\circ$$

$$= \frac{\sqrt{3}}{4}\times\frac{1}{2}(2\sin 80^\circ \cos 20^\circ) - \frac{\sqrt{3}}{4}\times\frac{1}{2}\sin 80^\circ$$

$$= \frac{\sqrt{3}}{8}(\sin 100^\circ + \sin 60^\circ) - \frac{\sqrt{3}}{8}\sin(180^\circ - 100^\circ)$$

$$= \frac{\sqrt{3}}{8}\sin 100^\circ + \frac{\sqrt{3}}{8}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\sin 100^\circ = \frac{3}{16} = \text{RHS}$$

(vi) $\text{LHS} = \sin 20^\circ \sin 40^\circ \sin 80^\circ$

$$= \frac{1}{2}(2\sin 20^\circ \sin 40^\circ)\sin 80^\circ$$

$$= \frac{1}{2}(\cos 20^\circ - \cos 60^\circ)\sin 80^\circ$$

$$= \frac{1}{2}\cos 20^\circ \sin 80^\circ - \frac{1}{2}\cos 60^\circ \sin 80^\circ$$

$$= \frac{1}{4}(2\sin 80^\circ \cos 20^\circ) - \frac{1}{4}\sin 80^\circ$$

$$= \frac{1}{4}(\sin 100^\circ + \sin 60^\circ) - \frac{1}{4}\sin(180^\circ - 100^\circ)$$

$$= \frac{1}{4}\sin 100^\circ + \frac{1}{4}\times\frac{\sqrt{3}}{2} - \frac{1}{4}\sin 100^\circ = \frac{\sqrt{3}}{8} = \text{RHS}$$

(vii) $\text{LHS} = \cos 130^\circ + \cos 110^\circ + \cos 10^\circ$

$$= 2\cos\frac{120^0}{2}\cos\frac{100^0}{2} + \cos 130^\circ$$

2.3 Transformation of Sums and Products

We already have the following results-

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ (i)

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ (ii)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ (iii)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ (iv)

(i) + (iii) $\Rightarrow$ $\mathbf{\sin(A+B) + \sin(A-B) = 2\sin A \cos B}$ (v)

(i) − (iii) $\Rightarrow$ $\mathbf{\sin(A+B) - \sin(A-B) = 2\cos A \sin B}$ (vi)

(ii) + (iv) $\Rightarrow$ $\mathbf{\cos(A+B) + \cos(A-B) = 2\cos A \cos B}$ (vii)

(iv) − (ii) $\Rightarrow$ $\mathbf{\cos(A-B) - \cos(A+B) = 2\sin A \sin B}$ (viii)

(ii) − (iv) $\Rightarrow$ $\mathbf{\cos(A+B) - \cos(A-B) = -2\sin A \sin B}$

2.3.1 Expression of sum and difference as product:

Let $\mathbf{A+B=C}$ and $\mathbf{A-B=D}$ $\therefore$ $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

From (v) we get $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (ix)

From (vi) we get $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$ (x)

From (vii) we get $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (xi)

From (viii) we get $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$ (xii)

Worked out examples:

Ex 1. (i) Express as sum: a. $2\cos 2\theta \sin 5\theta$ b. $2\cos\dfrac{17}{2}\theta\cos\dfrac{7}{2}\theta$

(ii) Find the value of $\sin 75^\circ \sin 15^\circ$

Soln: (i) a. $2\cos 2\theta \sin 5\theta = \sin(2\theta + 5\theta) - \sin(2\theta - 5\theta) = \sin 7\theta - \sin(-3\theta)$
$= \sin 7\theta + \sin 3\theta$

b. $2\cos\dfrac{17}{2}\theta\cos\dfrac{7}{2}\theta = \cos(\dfrac{17}{2}\theta + \dfrac{7}{2}\theta) + \cos(\dfrac{17}{2}\theta - \dfrac{7}{2}\theta) = \cos 12\theta + \cos 5\theta$

(ii) $\sin 75^\circ \sin 15^\circ = \dfrac{1}{2}[2\sin 75^\circ \sin 15^\circ] = \dfrac{1}{2}[\cos(75^\circ - 15^\circ) - \cos(75^\circ + 15^\circ)]$

$= \dfrac{1}{2}(\cos 60^\circ - \cos 90^\circ) = \dfrac{1}{2}(\dfrac{1}{2} - 0) = \dfrac{1}{4}$

Ex 2. (i) Express as product: a. **[QP 2011]** $\sin 6\theta + \sin 3\theta$

b. **[QP 2009]** $\sin 80^\circ - \cos 70^\circ$

(ii) **[QP 2013]** Evaluate $\cos 105^\circ - \cos 15^\circ$

Soln: (i) a. $\sin 6\theta + \sin 3\theta = 2\sin\dfrac{6\theta + 3\theta}{2}\cos\dfrac{6\theta - 3\theta}{2} = 2\sin\dfrac{9\theta}{2}\cos\dfrac{3\theta}{2}$

b. $\sin 80^\circ - \cos 70^\circ = \sin 80^\circ - \cos(90^\circ - 20^\circ) = \sin 80^\circ - \sin 20^\circ$

$= 2\cos\dfrac{80^\circ + 20^\circ}{2}\sin\dfrac{80^\circ - 20^\circ}{2} = 2\cos 50^\circ \sin 30^\circ$

(ii) $\cos 105^\circ - \cos 15^\circ = 2\sin\dfrac{105^\circ + 15^\circ}{2}\sin\dfrac{15^\circ - 105^\circ}{2}$

$= 2\sin 60^\circ \sin(-45^\circ) = 2\dfrac{\sqrt{3}}{2}(-\dfrac{1}{\sqrt{2}}) = -\dfrac{\sqrt{3}}{\sqrt{2}}$

Ex 3. Prove that (i) $\sin 25^\circ + \cos 25^\circ = \sqrt{2}\cos 20^\circ$

(ii) **[QP 2011]** $\cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0$

(iii) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

(iv) **[QP 2009, 2014, 2015]** $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$

(v) **[QP 2010]** $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$

(vi) **[QP 2013, 2015]** $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \dfrac{\sqrt{3}}{8}$

(vii) **[QP 2009, 2013, 2017]** $\cos 130^\circ + \cos 110^\circ + \cos 10^\circ = 0$

(viii) **[QP 2011]** $\cos(120^\circ + A) + \cos(120^\circ - A) + \cos A = 0$

(ix) **[QP 2014]** $\tan A + \cot A = 2\operatorname{cosec} 2A$

(x) **[QP 2014]** $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$

Proof: (i) LHS $= \sin 25^\circ + \cos 25^\circ = \sin(90^\circ - 65^\circ) + \cos 25^\circ = \cos 65^\circ + \cos 25^\circ$

Now dividing by 2sin$A$ sin$B$ we get $\cot A = \dfrac{n-1}{n+1}\cot B$

Exercise 2.2

1. Find the value

(i) $\sec(-75^\circ)$ (ii) $\cot(-105^\circ)$ (iii) $\cos(105^\circ)$

(iv) $\cot(165^\circ)$ (v) $\sec 255^\circ$ (vi) $\cos(285^\circ)$ (vii) $\cos(195^\circ)$

Ans.(i) $\dfrac{2\sqrt{2}}{\sqrt{3}-1}$ (ii) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ (iii) $\dfrac{1-\sqrt{3}}{2\sqrt{2}}$

(iv) $-\dfrac{1+\sqrt{3}}{\sqrt{3}-1}$ (v) $-\sqrt{2}(1+\sqrt{3})$ (vi) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (vii) $-\dfrac{1+\sqrt{3}}{2\sqrt{2}}$

2. Prove that $\sin A + \cos A = \sqrt{2}\cos(45^\circ - A)$

3. Prove that $\dfrac{\sin(\gamma-\delta)}{\sin\gamma\sin\delta} = \cot\delta - \cot\gamma$

4. Prove that (i) $\tan(45^\circ - \theta) = \dfrac{1-\tan\theta}{1+\tan\theta}$ (ii) $\tan(45^\circ + \theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$

5. Prove that $\sin(270^\circ + \theta) = -\cos\theta$

6. Prove that $\cos A - \sin A = \sqrt{2}\sin(45^\circ - A) = \sqrt{2}\cos(45^\circ + A)$

7. Prove that $\cos 65^\circ + \sin 65^\circ = \sqrt{2}\cos 20^\circ$

8. Prove that $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$

9. Prove that (i) $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \tan 55^\circ$ (ii) $\dfrac{\cos 18^\circ - \sin 18^\circ}{\cos 18^\circ + \sin 18^\circ} = \tan 27^\circ$

10. If $A+B+C = \dfrac{\pi}{2}$ then prove that $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$

11. If $A+B+C = \pi$ and $\cos A = \cos B\cos C$ then prove that

(a) $\tan A = \tan B + \tan C$ (b) $\cot B\cot C = \dfrac{1}{2}$

12. Prove that $\dfrac{\sin(B-C)}{\cos B\cos C} = \tan B - \tan C$

13. If $\tan\alpha = \dfrac{7}{8}$ and $\tan\beta = \dfrac{5}{7}$ then find the value of $\cot(\alpha-\beta)$. Ans: $\dfrac{91}{9}$

14. If $\sin\alpha = \dfrac{2}{3}$ and $\cos\beta = \dfrac{3}{5}$ then find the value of $\sin(\alpha+\beta)$ Ans. $\dfrac{6+4\sqrt{5}}{15}$

15. If $\sin(\alpha-\beta) = \dfrac{5}{13}$ and $\cos(\alpha+\beta) = \dfrac{4}{5}$ then find the value of $\tan 2\alpha$

Ans. $\dfrac{56}{33}$

16. Prove that

(i) $\tan 16^\circ + \tan 29^\circ + \tan 16^\circ \tan 29^\circ = 1$

(ii) $(1 - \tan 55^\circ)\tan 100^\circ = 1 + \tan 55^\circ$

17. Prove that $1 + \cot\alpha\tan\beta = 0$ if $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

18. If $\alpha+\beta = \dfrac{\pi}{4}$ prove that $(\cot\alpha - 1)(\cot\beta - 1) = 2$

* * *

$= \sin^2A - \sin^2A\sin^2B - \sin^2B + \sin^2A\sin^2B = \sin^2A - \sin^2B$

And $\sin^2A - \sin^2B = (1-\cos^2A) - (1-\cos^2B) = \cos^2B - \cos^2A$. Hence the result

(vi) $\cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2A\cos^2B - \sin^2A\sin^2B = \cos^2A(1-\sin^2B) - (1-\cos^2A)\sin^2B$

$= \cos^2A - \cos^2A\sin^2B - \sin^2B + \cos^2A\sin^2B = \cos^2A - \sin^2B$

And $\cos^2A - \sin^2B = (1-\sin^2A) - (1-\cos^2B) = \cos^2B - \sin^2A$. Hence the result

(vii) $\tan(A+B)\tan(A-B) = \dfrac{\sin(A+B)\sin(A-B)}{\cos(A+B)\cos(A-B)} = \dfrac{\sin^2A - \sin^2B}{\cos^2A - \sin^2B}$ Proved.

(viii) LHS $= \sin(45^o+\theta)\sin(45^o-\theta) = \sin^2 45^o - \sin^2\theta = \dfrac{1}{2} - \sin^2\theta =$ RHS

(ix) LHS $= \tan(45^o+\theta)\tan(45^o-\theta) = \dfrac{\tan 45^o - \tan\theta^o}{1+\tan 45^o\tan\theta^o}\cdot\dfrac{\tan 45^o + \tan\theta^o}{1-\tan 45^o\tan\theta^o}$

$= \dfrac{1-\tan\theta^o}{1+\tan\theta^o}\cdot\dfrac{1+\tan\theta^o}{1-\tan\theta^o} = 1 =$ RHS

(x) LHS $= \dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} + \dfrac{\sin(A-B)}{\cos A\cos B}$

$= \dfrac{\sin B\cos C - \cos B\sin C}{\cos B\cos C} + \dfrac{\sin C\cos A - \cos C\sin A}{\cos C\cos A}$

$+ \dfrac{\sin A\cos B - \cos A\sin B}{\cos A\cos B}$

$= \tan B - \tan C + \tan C - \tan A + \tan A - \tan B = 0 =$ RHS

Ex 5.(i)[QP 2009, 2010, 2014, 2015] If $A+B=45^o$ prove that $(1+\tan A)(1+\tan B)=2$

(ii) [QP 2013] Prove that $(1+\tan 30^o)(1+\tan 15^o)=2$

(iii) Prove that $\tan 26^o + \tan 19^o + \tan 26^o\tan 19^o = 1$

(iv) [QP 2011, 2014] Prove that $\tan 27^o + \tan 18^o + \tan 27^o\tan 18^o = 1$

Proof: (i) Given, $A+B=45^o$

Now, $\tan(A+B) = \tan 45^o \Rightarrow \dfrac{\tan A + \tan B}{1-\tan A\tan B} = \tan\dfrac{\pi}{4} = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A\tan B \Rightarrow \tan A + \tan B + \tan A\tan B = 1$

Now, $(1+\tan A)(1+\tan B) = 1 + \tan A + \tan B + \tan A\tan B = 1+1 = 2$ proved.

(ii) $\tan(30^o+15^o) = \tan 45^o \Rightarrow \dfrac{\tan 30^o + \tan 15^o}{1-\tan 30^o\tan 15^o} = 1$

$\Rightarrow \tan 30^o + \tan 15^o + \tan 30^o\tan 15^o = 1$

Now $(1+\tan 30^o)(1+\tan 15^o) = 1 + \tan 30^o + \tan 15^o + \tan 30^o\tan 15^o = 1+1 = 2$

(iii) $\tan(26^o+19^o) = \tan 45^o \Rightarrow \dfrac{\tan 26^o + \tan 19^o}{1-\tan 26^o\tan 19^o} = 1$

$\Rightarrow \tan 26^o + \tan 19^o + \tan 26^o\tan 19^o = 1$

(iv) $\tan(27^o+18^o) = \tan 45^o \Rightarrow \dfrac{\tan 27^o + \tan 18^o}{1-\tan 27^o\tan 18^o} = 1$

$\Rightarrow \tan 27^o + \tan 18^o + \tan 27^o\tan 18^o = 1$

Ex 6.(i)[QP 2011, 2013] If $\tan\theta = \dfrac{a}{b}$ prove that $\dfrac{a\sin\theta - b\cos\theta}{a\sin\theta + b\cos\theta} = \dfrac{a^2-b^2}{a^2+b^2}$

(ii)[QP 2013] Show that $\cos(60^o-A)\cos(30^o-B) - \sin(60^o-A)\sin(30^o-B) = \sin(A+B)$

(iii)[QP 2014] If $\sin(A+B) = n\sin(A-B)$ then prove that $\cot A = \dfrac{n-1}{n+1}\cot B$

Proof: (i) LHS $= \dfrac{a\sin\theta - b\cos\theta}{a\sin\theta + b\cos\theta} = \dfrac{\dfrac{a\sin\theta - b\cos\theta}{b\cos\theta}}{\dfrac{a\sin\theta + b\cos\theta}{b\cos\theta}} = \dfrac{\dfrac{a\sin\theta}{b\cos\theta} - 1}{\dfrac{a\sin\theta}{b\cos\theta} + 1}$

$= \dfrac{\dfrac{a}{b}\tan\theta - 1}{\dfrac{a}{b}\tan\theta + 1} = \dfrac{\dfrac{a}{b}\cdot\dfrac{a}{b} - 1}{\dfrac{a}{b}\cdot\dfrac{a}{b} + 1} = \dfrac{a^2-b^2}{a^2+b^2} =$ RHS

(ii) LHS $= \cos(60^o-A)\cos(30^o-B) - \sin(60^o-A)\sin(30^o-B)$

$= \cos\{(60^o-A)+(30^o-B)\} = \cos\{90^o-(A+B)\} = \sin(A+B) =$ RHS

(iii) Given $\sin(A+B) = n\sin(A-B)$

$\Rightarrow \dfrac{\sin(A+B)}{\sin(A-B)} = n \Rightarrow \dfrac{\sin(A+B) - \sin(A-B)}{\sin(A+B) + \sin(A-B)} = \dfrac{n-1}{n+1}$

$\Rightarrow 2\cos A\sin B = \dfrac{n-1}{n+1}\, 2\sin A\cos B$

$$=\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\theta\tan\frac{\pi}{4}}=\tan(\frac{\pi}{4}+\theta)=\text{LHS}$$

(iv) $\text{LHS}=\tan(\frac{\pi}{4}+\frac{\theta}{2})=\dfrac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}=\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

$$=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}=\frac{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}$$

$$=\frac{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})^2}{(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})}=\frac{(1+2\cos\frac{\theta}{2}\sin\frac{\theta}{2})}{\cos\theta}=\frac{(1+\sin\theta)}{\cos\theta}$$

$$=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta=\text{RHS}$$

(v) $\text{LHS}=(\cos^2\theta)^3-(\sin^2\theta)^3=(\cos^2\theta-\sin^2\theta)^3+3\cos^2\theta\sin^2\theta(\cos^2\theta-\sin^2\theta)$

$$=(\cos2\theta)^3+3\times\frac{1}{4}\times4\cos^2\theta\sin^2\theta(\cos2\theta)$$

$$=(\cos2\theta)^3+3\times\frac{1}{4}\sin^22\theta(\cos2\theta)$$

$$=\cos2\theta(\cos^22\theta+3\times\frac{1}{4}\sin^22\theta)$$

$$=\cos2\theta\{\cos^22\theta+(1-\frac{1}{4})\sin^22\theta\}$$

$$=\cos2\theta(\cos^22\theta+\sin^22\theta-\frac{1}{4}\sin^22\theta)$$

$$=\cos2\theta\,(1-\frac{1}{4}\sin^22\theta)=\text{RHS}$$

(vi) $\text{LHS}=\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)=(\cos^2\theta-\sin^2\theta)\times1$
$=\cos2\theta=\text{RHS}$

(vii) $\text{LHS}=\dfrac{\sin\theta+\sin2\theta}{1+\cos\theta+\cos2\theta}=\dfrac{\sin\theta+2\sin\theta\cos\theta}{1+\cos\theta+2\cos^2\theta-1}$

$$=\frac{\sin\theta(1+2\cos\theta)}{\cos\theta+2\cos^2\theta}=\frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)}=\frac{\sin\theta}{\cos\theta}=\tan\theta=\text{RHS}$$

(viii) $\text{LHS}=\sin3A=\sin(2A+A)$
$=\sin A\cos2A+\cos A\sin2A$
$=\sin A(1-2\sin^2A)+\cos A\,2\sin A\cos A$
$=\sin A-2\sin^3A+2\sin A\cos^2A$
$=\sin A-2\sin^3A+2\sin A(1-\sin^2A)$
$=\sin A-2\sin^3A+2\sin A-2\sin^3A$
$=3\sin A-4\sin^3A=\text{RHS}$

(ix) $\text{LHS}=\cos3A=\cos(2A+A)$
$=\cos A\cos2A-\sin A\sin2A$
$=\cos A(2\cos^2A-1)-\sin A\,2\sin A\cos A$
$=2\cos^3A-\cos A-2\sin^2A\cos A$
$=2\cos^3A-\cos A-2(1-\cos^2A)\cos A$
$=2\cos^3A-\cos A-2\cos A+2\cos^3A$
$=4\cos^3A-3\cos A=\text{RHS}$

(x) $\text{LHS}=\tan3A=\tan(A+2A)$

$$=\frac{\tan A+\tan2A}{1-\tan A\tan2A}$$

$$=\frac{\tan A+\frac{2\tan A}{1-\tan^2A}}{1-\tan A\frac{2\tan A}{1-\tan^2A}}=\frac{\frac{\tan A-\tan^3A+2\tan A}{1-\tan^2A}}{1-\frac{2\tan^2A}{1-\tan^2A}}=\frac{3\tan A-\tan^3A}{1-3\tan^2A}$$

$=\text{RHS}$

(xi) $\text{LHS}=\sin4A=2\sin2A\cos2A$

$$=2\frac{2\tan A}{1+\tan^2A}\cdot\frac{1-\tan^2A}{1+\tan^2A}=\frac{4\tan A(1-\tan^2A)}{(1+\tan^2A)^2}=\text{RHS}$$

(xii) $\text{LHS}==\dfrac{\sin A+\sin2A+\sin3A}{\cos A+\cos2A+\cos3A}$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = (\frac{24}{25})/(\frac{-7}{25}) = -\frac{24}{7}$$

(ii) $a\cos 2\theta + b\sin 2\theta = a\dfrac{1-\tan^2\theta}{1+\tan^2\theta} + b\dfrac{2\tan\theta}{1+\tan^2\theta} = a\dfrac{1-(\frac{b}{a})^2}{1+(\frac{b}{a})^2} + b\dfrac{2\frac{b}{a}}{1+(\frac{b}{a})^2}$

$= a.\dfrac{a^2-b^2}{a^2+b^2} + b.\dfrac{2ab}{a^2+b^2} = \dfrac{a^3 - a\,b^2 + 2ab^2}{a^2+b^2} = \dfrac{a^3+ab^2}{a^2+b^2} = \dfrac{a(a^2+b^2)}{a^2+b^2} = a$ proved

Ex 2. Prove that (i) $\cot A = \dfrac{\sin 2A}{1-\cos 2A}$

(ii) **[QP 2010]** $\tan A = \dfrac{\sin 2A}{1+\cos 2A}$

(iii) $\tan(\dfrac{\pi}{4}+\theta) = \dfrac{1+\sin 2\theta}{\cos 2\theta}$

(iv) **[QP 2009, 2013]** $\tan(\dfrac{\pi}{4}+\dfrac{\theta}{2}) = \sec\theta + \tan\theta$

(v) **[QP 2009]** $\cos^6\theta - \sin^6\theta = \cos 2\theta\,(1-\dfrac{1}{4}\sin^2 2\theta)$

(vi) **[QP 2011]** $\cos^4\theta - \sin^4\theta = \cos 2\theta$

(vii) **[QP 2011]** $\dfrac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

(viii) **[QP 2017]** $\sin 3A = 3\sin A - 4\sin^3 A$

(ix) **[QP 2009, 2011]** $\cos 3A = 4\cos^3 A - 3\cos A$

(x) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

(xi) $\sin 4A = \dfrac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2}$

(xii) **[QP 2011]** $\dfrac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A$

(xiii) $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$

(xiv) **[QP 2011]** $\dfrac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \cot\theta$

(xv) **[QP 2014]** $\dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

(xvi) **[QP 2014]** $\sec x = \dfrac{2}{\sqrt{2+\sqrt{2+2\cos 4x}}}$

(xvii) **[QP 2015]** $\dfrac{\sin\theta - \sqrt{1+\sin 2\theta}}{\cos\theta - \sqrt{1+\sin 2\theta}} = \cot\theta$

(xviii) $16\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15} = 1$

(xix) $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \dfrac{1}{8}$

(xx) $\sin^2 42^\circ - \cos^2 78^\circ = \dfrac{\sqrt{5}+1}{8}$

Proof: (i) RHS $= \dfrac{\sin 2A}{1-\cos 2A} = \dfrac{2\sin A\cos A}{1-(1-2\sin^2 A)} = \dfrac{2\sin A\cos A}{2\sin^2 A} = \dfrac{\cos A}{\sin A}$

$= \cot A$ = LHS

(ii) RHS $= \dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{1+(2\cos^2 A - 1)} = \dfrac{2\sin A\cos A}{2\cos^2 A} = \dfrac{\sin A}{\cos A}$

$= \tan A$ = LHS

(iii) RHS $= \dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{(\sin\theta+\cos\theta)^2}{\cos^2\theta - \sin^2\theta} = \dfrac{(\sin\theta+\cos\theta)}{\cos\theta - \sin\theta} = \dfrac{1+\tan\theta}{1-\tan\theta}$ (Divide by $\cos\theta$)

2.4 Multiple and Submultiple Angles

We have in previous chapter that

Addition formula of sine and cosine for compound angles (A+B) are

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Putting A = B in (i) we have

$$\sin(A+A) = \sin A\cos A + \cos A\sin A$$

$$\Rightarrow \sin 2A = 2\sin A\cos A \quad \text{.......(1)}$$

Again putting A = B in (ii) we have

$$\cos(A+A) = \cos A\cos A - \sin A\sin A$$

$$\Rightarrow \cos 2A = \cos^2 A - \sin^2 A \quad \text{.......(2)}$$

$$\Rightarrow \cos 2A = \cos^2 A - (1-\cos^2 A)$$

$$= 2\cos^2 A - 1 \quad \text{.......(3)}$$

$$= 2(1-\sin^2 A) - 1 = 1 - 2\sin^2 A \quad \text{.......(4)}$$

From (3) $\cos 2A = 2\cos^2 A - 1$

$$\therefore 1 + \cos 2A = 2\cos^2 A \quad \text{.......(5)}$$

again from (4) $1 - \cos 2A = 2\sin^2 A$ ....(6)

Also we have from previous chapter that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

And putting A = B in above, we have

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A\tan A} \Rightarrow \tan 2A = \frac{2\tan A}{1-\tan^2 A} \quad \text{.......(7)}$$

In a similar way $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B} \Rightarrow \cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$ ....(8)

Note: 1. $1+\sin 2A = 1 + 2\sin A\cos A$

$$= \sin^2 A + \cos^2 A + 2\sin A\cos A$$

$$1+\sin 2A = (\sin A + \cos A)^2$$

2. $1 - \sin 2A = 1 - 2\sin A\cos A$

$$= \sin^2 A + \cos^2 A - 2\sin A\cos A$$

$$1 - \sin 2A = (\sin A - \cos A)^2$$

3. $\sin 2A = 2\sin A\cos A = \dfrac{2\sin A\cos A}{\sin^2 A + \cos^2 A}$

$$= \frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\cos^2 A}} = \frac{\dfrac{2\sin A}{\cos A}}{\dfrac{\sin^2 A}{\cos^2 A}+1} = \frac{2\tan A}{1+\tan^2 A}$$

4. $\cos 2A = \cos^2 A - \sin^2 A = \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$

$$= \frac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \frac{1 - \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{1-\tan^2 A}{1+\tan^2 A}$$

2.4.1 Corollary : (i) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

(ii) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2} = 2\cos^2\dfrac{A}{2} - 1 = 1 - 2\sin^2\dfrac{A}{2}$

(iii) $\tan A = \dfrac{2\tan\dfrac{A}{2}}{1-\tan^2\dfrac{A}{2}}$

Worked out examples :

Ex 1. (i) If $\cos\theta = \dfrac{3}{5}$ then find the value of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

(ii) **[QP 2009, 2015]** If $\tan\theta = \dfrac{b}{a}$, prove that $a\cos 2\theta + b\sin 2\theta = a$

Soln: (i) We know $\sin 2\theta = 2\sin\theta\cos\theta$

Given $\cos\theta = \dfrac{3}{5}$, $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(\frac{3}{5})^2} = \dfrac{4}{5}$

Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\dfrac{4}{5}\times\dfrac{3}{5} = \dfrac{24}{25}$

$= 2\cos 60^\circ \cos 50^\circ + \cos 130^\circ = 2\times\frac{1}{2}\cos 50^\circ + \cos(180^\circ - 50^\circ)$

$= \cos 50^\circ - \cos 50^\circ = 0 = \text{RHS}$

(viii) LHS $= \cos(120^\circ + A) + \cos(120^\circ - A) + \cos A$

$= 2\cos 120^\circ \cos A + \cos A$

$= 2\cos(180^\circ - 60^\circ)\cos A + \cos A$

$= 2(-\cos 60^\circ)\cos A + \cos A = 2\times(-\frac{1}{2})\cos A + \cos A$

$= -\cos A + \cos A = 0 = \text{RHS}$

(ix) LHS $= \tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$

$= \frac{1}{\cos A \sin A} = \frac{2}{\cos A \sin A + \sin A \cos A} = \frac{2}{\sin(A+A)}$

$= 2\operatorname{cosec}(A+A) = 2\operatorname{cosec} 2A = \text{RHS}$

(x) LHS $= \frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \frac{2\cos 4A \sin A}{2\cos 4A \cos A} = \tan A = \text{RHS}$

Ex 4. Prove that (i) $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan\frac{A+B}{2}$

(ii) $\tan\frac{x+y}{2} = \frac{3}{4}$ if $\cos x + \cos y = \frac{1}{3}$ and $\sin x + \sin y = \frac{1}{4}$

Proof: (i) LHS $= \dfrac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}} = \tan\frac{A+B}{2} = \text{RHS}$

(ii) $\frac{\sin x + \sin y}{\cos x + \cos y} = \dfrac{2\sin\frac{x+y}{2}\cos\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}} = \tan\frac{x+y}{2} = (\frac{1}{4})/(\frac{1}{3}) = \frac{3}{4}$ proved.

Ex 5. Express 4cosAcosBcosC as the sum of four cosines.

Soln: 4cosAcosBcosC

$= 2\cos A\,(2\cos B\cos C)$

$= 2\cos A\,[\cos(B+C) + \cos(B-C)]$

$= 2\cos A\cos(B+C) + 2\cos A\cos(B-C)$

$= \cos\{A+(B+C)\} + \cos\{A-(B+C)\} + \cos\{A+(B-C)\} + \cos\{A-(B-C)\}$

$= \cos(A+B+C) + \cos(A-B-C) + \cos(A+B-C) + \cos(A-B+C)$

Exercise 2.3

1. Express as sum:
   (i) $2\sin 6\theta \sin 8\theta$ (ii) $2\cos 5\theta \cos 3\theta$ (iii) $2\sin 4\theta \cos 8\theta$ (iv) $2\cos 8\theta \sin 2\theta$
   Ans: (i) $\cos 2\theta - \cos 14\theta$ (ii) $\cos 8\theta + \cos 2\theta$
   (iii) $\sin 12\theta - \sin 4\theta$ (iv) $\sin 10\theta - \sin 6\theta$
2. Express as product:
   (i) $\sin 2\theta - \sin 8\theta$ (ii) $\cos 12\theta - \cos 8\theta$ (iii) $\cos 8\theta - \cos 4\theta$
   Ans. (i) $-2\cos 5\theta \sin 3\theta$; (ii) $-2\sin 10\theta \sin 2\theta$; (iii) $-2\sin 6\theta \sin 2\theta$
3. Prove that: (i) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$
   (ii) $\sin 10^\circ \sin 50^\circ \sin 60^\circ \sin 70^\circ = \frac{\sqrt{3}}{16}$
4. Prove that (i) $\sin 18^\circ + \cos 18^\circ - \sqrt{2}\cos 27^\circ = 0$
   (ii) $\cos 52^\circ + \cos 68^\circ + \cos 172^\circ = 0$
5. Prove that $\cos 40^\circ + \cos 50^\circ + \cos 70^\circ + \cos 80^\circ = \cos 10^\circ + \cos 20^\circ$
6. Prove that $\sin 20^\circ + \sin 140^\circ - \cos 10^\circ = 0$
7. Show that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$
8. Prove that $\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{1}{\sqrt{3}}$
9. Prove that $\sec(\frac{\pi}{4} + \theta)\sec(\frac{\pi}{4} - \theta) = 2\sec 2\theta$
10. If $\cos\beta = k\cos\alpha$ then prove that $\tan\frac{\alpha-\beta}{2} = \frac{k-1}{k+1}\cot\frac{\alpha+\beta}{2}$

*****

$$\text{(ii)}\ \frac{b}{a}=\frac{\cos x+\cos y}{\sin x+\sin y}=\frac{2\cos\frac{(x+y)}{2}\cos\frac{(x-y)}{2}}{2\sin\frac{(x+y)}{2}\cos\frac{(x-y)}{2}}=\frac{\cos\frac{(x+y)}{2}}{\sin\frac{(x+y)}{2}}$$

$$\Rightarrow\frac{b^2}{a^2}=\frac{\cos^2\frac{(x+y)}{2}}{\sin^2\frac{(x+y)}{2}}$$

$$\Rightarrow\frac{b^2-a^2}{b^2+a^2}=\frac{\cos^2\frac{x+y}{2}-\sin^2\frac{x+y}{2}}{\cos^2\frac{x+y}{2}+\sin^2\frac{x+y}{2}}=\cos^2\frac{x+y}{2}-\sin^2\frac{x+y}{2}=\cos(x+y)$$

(iii) $\sec(x+y)+\sec(x-y)=2\sec x$

$$\Rightarrow\frac{1}{\cos(x+y)}+\frac{1}{\cos(x-y)}=\frac{2}{\cos x}$$

$$\Rightarrow\frac{\cos(x-y)+\cos(x+y)}{\cos(x+y)\cos(x-y)}=\frac{2}{\cos x}$$

$$\Rightarrow\frac{2\cos x\cos y}{\cos^2 x-\sin^2 y}=\frac{2}{\cos x}$$

$$\Rightarrow\cos^2 x\cos y=\cos^2 x-\sin^2 y$$

$$\Rightarrow\cos^2 x(1-\cos y)=\sin^2 y$$

$$\Rightarrow\cos^2 x=\frac{4\sin^2\frac{y}{2}\cos^2\frac{y}{2}}{2\sin^2\frac{y}{2}}=2\cos^2\frac{y}{2}$$

$$\Rightarrow\cos x=\sqrt{2}\cos\frac{y}{2}$$

Exercise 2.4

1. If $\tan\theta=t$ then prove that (i) $\cos 2\theta=\frac{1-t^2}{1+t^2}$

   (ii) $\sin 2\theta=\frac{2t}{1+t^2}$ (iii) $\tan 2\theta=\frac{2t}{1-t^2}$

2. Find the value of $\sin 2\theta, \cos 2\theta$ and $\tan 2\theta$

   (i) If $\sin\theta=\frac{4}{5}$, θ is in 2nd quadrant

   (ii) If $\cos\theta=-\frac{1}{2}$, θ is in 3rd quadrant

   Ans. (i) $\sin 2\theta=-\frac{24}{25}$, $\cos 2\theta=-\frac{7}{25}$, $\tan 2\theta=\frac{24}{7}$

   (ii) $\sin 2\theta=\frac{\sqrt{3}}{2}$, $\cos 2\theta=\frac{-1}{2}$, $\tan 2\theta=-\sqrt{3}$

3. If $\sin\alpha=-\frac{4}{5}$, α is in 3rd quadrant then find the value of $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$. Ans. $\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}$

4. (i) If $\cos(\alpha+\beta)=\frac{4}{5}$ and $\sin(\alpha-\beta)=\frac{5}{13}$ then find the value of $\sin 2\alpha$.

   (ii) If $2\cos\theta=a+\frac{1}{a}$ then find the value of $\cos 2\theta$.

   Ans. (i) $\frac{56}{65}$ (ii) $\frac{a^4+1}{2a^2}$

5. (i) Express $\cos 5A$ in terms of $\cos A$. (ii) Express $\cos 4A$ in terms of $\sin A$

   (iii) Express $\tan 4A$ in terms of $\tan A$

6. Prove that :

   (i) $\cot\theta-\tan\theta=2\cot 2\theta$ (ii) $\cos^4\theta+\sin^4\theta=\frac{1}{2}(1+\cos^2 2\theta)$

We have now $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

(iv) $\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\left(\dfrac{\sqrt{5}-1}{4}\right)^2}$

$$= \frac{\sqrt{16-(5+1-2\sqrt{5})}}{4} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(v) $\sin 36^\circ = 2\sin 18^\circ \cos 18^\circ = 2\,\dfrac{\sqrt{5}-1}{4}.\dfrac{\sqrt{10+2\sqrt{5}}}{4}$

$$= \frac{\sqrt{(6-2\sqrt{5})(10+2\sqrt{5})}}{8} = \frac{\sqrt{40-8\sqrt{5}}}{8} = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

(vi) $\cos 54^\circ = \cos(90^\circ-36^\circ) = \sin 36^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

(vii) $\tan 36^\circ = \dfrac{\sin 36^\circ}{\cos 36^\circ}$

$$\cos 36^\circ = 1-2\sin^2 18^\circ = 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{8-(6-2\sqrt{5})}{8}$$

$$= \frac{\sqrt{5}+1}{4}$$

Now $\tan 36^\circ = \dfrac{\sin 36^\circ}{\cos 36^\circ} = \left(\dfrac{\sqrt{10-2\sqrt{5}}}{4}\right)\left(\dfrac{4}{\sqrt{5}+1}\right) = \dfrac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$

(viii) $\sin 54^\circ = \sin(90^\circ-36^\circ) = \cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$

(ix) $\tan 72^\circ = \tan(90^\circ-18^\circ) = \cot 18^\circ = \dfrac{\cos 18^\circ}{\sin 18^\circ}$

$$= \left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)\left(\frac{4}{\sqrt{5}-1}\right) = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$$

(x) $\cot 18^\circ = \dfrac{\cos 18^\circ}{\sin 18^\circ} = \left(\dfrac{\sqrt{10+2\sqrt{5}}}{4}\right)\left(\dfrac{4}{\sqrt{5}-1}\right) = \dfrac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$

Ex 5. (i) Find the value of: a. $\cos\dfrac{\pi}{32}$ b. $\sin(67\tfrac{1}{2})^\circ$

(ii) If $\sin x+\sin y = a$ and $\cos x+\cos y = b$ then prove that

$$\cos(x+y) = \frac{b^2-a^2}{b^2+a^2}$$

(iii) If $\sec(x+y)+\sec(x-y) = 2\sec x$ prove that $\cos x = \sqrt{2}\cos\dfrac{y}{2}$

Soln: (i) a. Here we apply the formula $1+\cos 2A = 2\cos^2 A$. Equivalently

$$\cos A = \sqrt{\frac{1+\cos 2A}{2}}$$

Thus $\cos\dfrac{\pi}{8} = \sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}} = \dfrac{1}{\sqrt{2}}\sqrt{1+\dfrac{1}{\sqrt{2}}} = \dfrac{1}{2}\sqrt{2+\sqrt{2}}$

$$\cos\frac{\pi}{16} = \sqrt{\frac{1+\cos\frac{\pi}{8}}{2}} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{2}\sqrt{2+\sqrt{2}}} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

Lastly $\cos\dfrac{\pi}{32} = \sqrt{\dfrac{1+\cos\frac{\pi}{16}}{2}} = \dfrac{1}{\sqrt{2}}\sqrt{1+\dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}}$

$$= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

b. $\sin(67\tfrac{1}{2})^\circ = \sin[90^\circ-22(\tfrac{1}{2})^\circ] = \cos 22(\tfrac{1}{2})^\circ = \cos\dfrac{\pi}{8} = \dfrac{1}{2}\sqrt{2+\sqrt{2}}$

$$=\frac{1}{2}\times\frac{\sqrt{5}+1}{4}=\frac{\sqrt{5}+1}{8}=\text{RHS}$$

Ex 3. (i) **[QP 2009]** Express $1+\sin\theta$ as a perfect square.

(ii) Express $\cos 4A$ in terms of $\cos A$.

(iii) $\tan\alpha=\frac{m}{m+1}$, $\tan\beta=\frac{1}{2m+1}$ Show that $\alpha+\beta=\frac{\pi}{4}$

Soln: (i) $1+\sin\theta=1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\Rightarrow\ 1+\sin\theta=\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2$$

(ii) $\cos 4A=\cos 2(2A)=2\cos^2 2A-1=2(\cos 2A)^2-1$

$=2(2\cos^2 A-1)^2-1$

$=2(4\cos^4 A-4\cos^2 A+1)-1$

$=8\cos^4 A-8\cos^2 A+2-1=8\cos^4 A-8\cos^2 A+1$

(iii) $\tan\alpha=\frac{m}{m+1}$, $\tan\beta=\frac{1}{2m+1}$

We have $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{m+1}\frac{1}{2m+1}}$$

$$=\frac{m(2m+1)+(m+1)}{(2m+1)(m+1)-m}=\frac{2m^2+2m+1}{2m^2+2m+1}=1=\tan\frac{\pi}{4}$$

$$\therefore\tan(\alpha+\beta)=\tan\frac{\pi}{4}\Rightarrow\alpha+\beta=\frac{\pi}{4}$$

Ex 4. Find the value of:

(i) $\tan(7\frac{1}{2})^\circ$ (ii) $\sin(22\frac{1}{2})^\circ$ (iii) **[2017]** $\sin 18^\circ$ (iv) $\cos 18^\circ$

(v) $\sin 36^\circ$ (vi) $\cos 54^\circ$ (vii) $\tan 36^\circ$ (viii) $\sin 54^\circ$

(ix) $\tan 72^\circ$ (x) $\cot 18^\circ$

Soln: (i) $$\tan(7\frac{1}{2})^\circ=\frac{\sin 7\frac{1}{2}^\circ}{\cos 7\frac{1}{2}^\circ}=\frac{2\sin 7\frac{1}{2}^\circ\cos 7\frac{1}{2}^\circ}{2\cos 7\frac{1}{2}^\circ\cos 7\frac{1}{2}^\circ}=\frac{\sin 15^\circ}{1+\cos 15^\circ}$$

$$=\frac{\sin(45^\circ-30^\circ)}{1+\cos(45^\circ-30^\circ)}$$

$$=\frac{\sin 45^\circ\cos 30^\circ-\sin 30^\circ\cos 45^\circ}{1+\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ}$$

$$=\frac{\frac{1}{\sqrt2}\frac{\sqrt3}{2}-\frac{1}{\sqrt2}\frac{1}{2}}{1+\frac{1}{\sqrt2}\frac{\sqrt3}{2}+\frac{1}{\sqrt2}\frac{1}{2}}=\frac{\sqrt3-1}{1+\sqrt3+2\sqrt2}$$

(ii) $$\sin 22\frac{1}{2}^\circ=\sin\left(\frac{45}{2}\right)^\circ=\sqrt{\frac{1-\cos 45^\circ}{2}}$$

$$=\frac{1}{\sqrt2}\sqrt{1-\frac{1}{\sqrt2}}=\frac{1}{2}\sqrt{2-\sqrt2}$$

(iii) To find $\sin 18^\circ$ we proceed as follows:

Let $\theta=18^\circ$

$\Rightarrow 5\theta=90^\circ$

$\Rightarrow \sin 2\theta=\sin(90^\circ-3\theta)=\cos 3\theta$

$\Rightarrow 2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta=\cos\theta(4\cos^2\theta-3)$

$\Rightarrow 2\sin\theta=4\cos^2\theta-3\quad(\because\cos\theta\neq0)$

$\Rightarrow 4\cos^2\theta-3-2\sin\theta=0$

$\Rightarrow 4(1-\sin^2\theta)-3-2\sin\theta=0$

$\Rightarrow 4-4\sin^2\theta-3-2\sin\theta=0$

$\Rightarrow 4\sin^2\theta+2\sin\theta-1=0$

$$\Rightarrow\sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{\sqrt5-1}{4}\qquad 0<\theta<90^\circ\ \Rightarrow\sin\theta>0$$

$$= \frac{\sin A + 2\sin A\cos A + 3\sin A - 4\sin^3 A}{\cos A + 2\cos^2 A - 1 + 4\cos^3 A - 3\cos A}$$

$$= \frac{4\sin A + 2\sin A\cos A - 4\sin A(1-\cos^2 A)}{1(2\cos^2 A - 1) + 2\cos A(2\cos^2 A - 1)}$$

$$= \frac{4\sin A + 2\sin A\cos A - 4\sin A + 4\sin A\cos^2 A}{(2\cos^2 A - 1)(1 + 2\cos A)}$$

$$= \frac{2\sin A\cos A + 4\sin A\cos^2 A}{(2\cos^2 A - 1)(1 + 2\cos A)}$$

$$= \frac{2\sin A\cos A(1 + 2\cos A)}{(2\cos^2 A - 1)(1 + 2\cos A)} = \frac{2\sin A\cos A}{(2\cos^2 A - 1)} = \frac{\sin 2A}{\cos 2A} = \tan 2A = \text{RHS}$$

(xiii) $$\text{LHS} = \frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = \frac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$$

$$= \frac{4(\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ)}{2\sin 10^\circ\cos 10^\circ} = \frac{4\sin(30^\circ - 10^\circ)}{\sin 20^\circ} = \frac{4\sin 20^\circ}{\sin 20^\circ} = 4 = \text{RHS}$$

(xiv) $$\text{LHS} = \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta}$$

$$= \frac{1 + 2\sin\theta\cos\theta + 2\cos^2\theta - 1}{1 + 2\sin\theta\cos\theta - 1 + 2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta + 2\cos^2\theta}{2\sin\theta\cos\theta + 2\sin^2\theta} = \frac{2\cos\theta(\sin\theta + \cos\theta)}{2\sin\theta(\cos\theta + \sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$$

(xv) $$\text{LHS} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta}$$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})}{2\cos\frac{\theta}{2}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})} = \tan\frac{\theta}{2} = \text{RHS}$$

(xvi) $$\text{RHS} = \frac{2}{\sqrt{2 + \sqrt{2 + 2\cos 4x}}} = \frac{2}{\sqrt{2 + \sqrt{2 \times 2\cos^2 2x}}}$$

$$= \frac{2}{\sqrt{2 + 2\cos 2x}} = \frac{2}{\sqrt{2 \times 2\cos^2 x}} = \frac{2}{2\cos x} = \sec x = \text{LHS}$$

(xvii) $$\text{LHS} = \frac{\sin\theta - \sqrt{1 + \sin 2\theta}}{\cos\theta - \sqrt{1 + \sin 2\theta}} = \frac{\sin\theta - (\sin\theta + \cos\theta)}{\cos\theta - (\sin\theta + \cos\theta)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta$$

$= \text{RHS}$

(xviii) $$16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}$$

$$= 4(2\cos\frac{2\pi}{15}\cos\frac{8\pi}{15})(2\cos\frac{4\pi}{15}\cos\frac{14\pi}{15})$$

$$= 4(\cos\frac{10\pi}{15} + \cos\frac{6\pi}{15})(\cos\frac{18\pi}{15} + \cos\frac{10\pi}{15})$$

$$= 4\{\cos(\pi - \frac{\pi}{3}) + \cos(\frac{\pi}{2} - \frac{\pi}{10})\}\{\cos(\pi + \frac{2\pi}{10}) + \cos(\pi - \frac{\pi}{3})\}$$

$$= 4(-\frac{1}{2} + \sin 18^\circ)\{(-\cos 36^\circ) + (-\frac{1}{2})\}$$

$$= (1 - 2\sin 18^\circ)(1 + \cos 36^\circ) = (1 - 2 \times \frac{\sqrt{5} - 1}{4})(1 + 2 \times \frac{\sqrt{5} + 1}{4})$$ see Ex 4.

$$= \frac{3 - \sqrt{5}}{2} \times \frac{3 + \sqrt{5}}{2} = 1 = \text{RHS}$$

(xix) $\sin 12^\circ \sin 48^\circ \sin 54^\circ$

$$= \frac{1}{2}(\cos 36^\circ - \cos 60^\circ)\sin 54^\circ$$

$$= \frac{1}{2}(\frac{\sqrt{5} + 1}{4} - \frac{1}{2})\frac{\sqrt{5} + 1}{4} = \frac{1}{2} \times \frac{\sqrt{5} - 1}{4} \times \frac{\sqrt{5} + 1}{4} = \frac{1}{8} = \text{RHS}$$ see Ex 4.

(xx) $\sin^2 42^\circ - \cos^2 78^\circ$

$= -\cos 120^\circ \cos 36^\circ$ since $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

(xiv) LHS $= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

$= \cos\frac{B+C}{2} + \cos\frac{C+A}{2} + \cos\frac{A+B}{2}$ $\qquad (\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2})$

$= \cos X + \cos Y + \cos Z$ $\qquad (X = \frac{B+C}{2}, Y = \frac{C+A}{2}, Z = \frac{A+B}{2})$

$= 1 + 4\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$ $\qquad$ see Ex 1(vii) $\quad$ here $X+Y+Z=\pi$

$= 1 + 4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}$

$= 1 + 4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4} =$ RHS $\qquad$ proved.

(xv) From Cor (iii) of 2.5.5 we have

$$\tan(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}) = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}-\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{1-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{C}{2}\tan\frac{A}{2}-\tan\frac{A}{2}\tan\frac{B}{2}} = \tan\frac{\pi}{2}$$

$\Rightarrow 1-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{C}{2}\tan\frac{A}{2}-\tan\frac{A}{2}\tan\frac{B}{2}=0$

$\Rightarrow \tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$ proved.

Ex 2. If $A+B+C=\frac{\pi}{2}$ then prove that

(i) $\cot A+\cot B+\cot C=\cot A\cot B\cot C$

(ii) **[QP 2011]** $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

(iii) $\sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C=1$

Proof: (i) Given, $A+B+C=\frac{\pi}{2}$ $\therefore A+B=\frac{\pi}{2}-C$ $\therefore \cot(A+B)=\cot(\frac{\pi}{2}-C)$

$\Rightarrow \frac{\cot A\cot B-1}{\cot A+\cot B}=\tan C$

$\Rightarrow \frac{\cot A\cot B-1}{\cot A+\cot B}=\frac{1}{\cot C}$

$\Rightarrow \cot C(\cot A\cot B-1)=\cot A+\cot B$

$\Rightarrow \cot C\cot A\cot B-\cot C=\cot A+\cot B$

$\Rightarrow \cot C\cot A\cot B=\cot A+\cot B+\cot C$ $\qquad$ proved.

(ii) Given, $A+B+C=\frac{\pi}{2}$ $\Rightarrow A+B=\frac{\pi}{2}-C$ $\therefore \tan(A+B)=\tan(\frac{\pi}{2}-C)$

$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\cot C$

$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{1}{\tan C}$

$\Rightarrow \tan C(\tan A+\tan B)=1-\tan A\tan B$

$\Rightarrow \tan C\tan A+\tan C\tan B+\tan A\tan B=1$

$\Rightarrow \tan A\tan B+\tan B\tan C+\tan C\tan A=1$ $\qquad$ Proved.

(iii) LHS $=\sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C$

$=\frac{1}{2}(2\sin^2 A+2\sin^2 B+2\sin^2 C)+2\sin A\sin B\sin C$

$=\frac{1}{2}(1-\cos 2A+1-\cos 2B+1-\cos 2C)+2\sin A\sin B\sin C$

$=\frac{3}{2}-\frac{1}{2}(\cos 2A+\cos 2B+\cos 2C)+2\sin A\sin B\sin C$

$=\frac{3}{2}-\frac{1}{2}(1+4\sin A\sin B\sin C)+2\sin A\sin B\sin C$ $\quad$ see Ex 1(vii)

$=1=$ RHS $\qquad$ proved.

Ex 3. (i) If $A+B+C=\pi$, $\cos A=\cos B\cos C$ then prove that

a. **[QP 2009]** $\tan A=\tan B+\tan C$ $\quad$ b. **[QP 2014]** $\cot B\cot C=\frac{1}{2}$

(ii) If $A+B=C$ then prove that

(ix) LHS $= \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C$

$= \frac{1}{2}(2\cos^2 A + 2\cos^2 B + 2\cos^2 C) + 2\cos A\cos B\cos C$

$= \frac{1}{2}(1+\cos 2A + 1+\cos 2B + 1+\cos 2C) + 2\cos A\cos B\cos C$

$= \frac{3}{2} + \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C) + 2\cos A\cos B\cos C$

$= \frac{3}{2} + \frac{1}{2}(-4\cos A\cos B\cos C - 1) + 2\cos A\cos B\cos C$ see Ex 1(vi)

$= \frac{3}{2} - \frac{1}{2} - 2\cos A\cos B\cos C + 2\cos A\cos B\cos C = 1 =$ RHS proved.

(x) LHS $= \cos A + \cos B - \cos C$

$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \cos C$

$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 1 + 2\sin^2\frac{C}{2}$ $[\cos\frac{A+B}{2} = \cos(\frac{\pi}{2} - \frac{C}{2}) = \sin\frac{C}{2}]$

$= 2\sin\frac{C}{2}[\cos\frac{A-B}{2} + \sin\frac{C}{2}] - 1$

$= 2\sin\frac{C}{2}[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}] - 1$

$= -1 + 2\sin\frac{C}{2}\, 2\cos\frac{A}{2}\cos\frac{B}{2} = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} =$ RHS proved

(xi) LHS $= \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$

$= \sin\frac{B+C}{2} + \sin\frac{C+A}{2} + \sin\frac{A+B}{2}$ $(\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2})$

$= \sin X + \sin Y + \sin Z$ $(X = \frac{B+C}{2}, Y = \frac{C+A}{2}, Z = \frac{A+B}{2})$

$= 4\cos\frac{X}{2}\cos\frac{Y}{2}\cos\frac{Z}{2}$ see Ex 1(v) here $X + Y + Z = \pi$

$= 4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}$

$= 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\cos\frac{\pi - C}{4} =$ RHS proved.

(xii) LHS $= \sin(B+2C) + \sin(C+2A) + \sin(A+2B)$

$= \sin(\pi - A + C) + \sin(\pi - B + A) + \sin(\pi - C + B)$

$= \sin(A-C) + \sin(B-A) + \sin(C-B)$

$= 2\sin(\frac{B-C}{2})\cos(\frac{A-C}{2} - \frac{B-A}{2}) - 2\sin(\frac{B-C}{2})\cos(\frac{B-C}{2})$

$= 2\sin(\frac{B-C}{2})[\cos(\frac{A-C}{2} - \frac{B-A}{2}) - \cos(\frac{A-C}{2} + \frac{B-A}{2})]$

$= 2\sin\frac{B-C}{2}\, 2\sin\frac{A-C}{2}\sin\frac{B-A}{2}$

$= 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2} =$ RHS proved.

(xiii) LHS $= \frac{\cos A}{\sin B\sin C} + \frac{\cos B}{\sin A\sin C} + \frac{\cos C}{\sin A\sin B}$

$= \frac{\cos(\pi - (B+C))}{\sin B\sin C} + \frac{\cos(\pi - (C+A))}{\sin C\sin A} + \frac{\cos(\pi - (A+B))}{\sin A\sin B}$

$= \frac{-(\cos B\cos C - \sin B\sin C)}{\sin B\sin C} + \frac{-(\cos C\cos A - \sin C\sin A)}{\sin C\sin A} + \frac{-(\cos A\cos B - \sin A\sin B)}{\sin A\sin B}$

$= 3 - (\cot B\cot C + \cot C\cot A + \cot A\cot B)$

$= 3 - 1$ see Ex 1.(ii)

$= 2 =$ RHS proved.

$\Rightarrow \tan(A+B) = \tan(\pi - C)$

$\Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow \tan A + \tan B = -\tan C(1 - \tan A \tan B)$

$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

Multiply both side by cotA cotB cotC we have,

$\cot A \cot B \cot C(\tan A + \tan B + \tan C) = \cot A \cot B \cot C(\tan A.\tan B.\tan C)$

$\Rightarrow \cot A \cot B + \cot B \cot C + \cot A \cot C = 1$ proved.

(iii) LHS $= \sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C \cos C$

$= 2\sin C \cos(A-B) + 2\sin C \cos C$ $\quad [\sin(A+B) = \sin(\pi - C) = \sin C]$

$= 2\sin C[\cos(A-B) + \cos C)]$

$= 2\sin C[\cos(A-B) - \cos(A+B)]$ $\quad [\cos C = \cos[\pi - (A+B)] = -\cos(A+B)]$

$= 2\sin C[2\sin A \sin B] = 4\sin A \sin B \sin C = \text{RHS}$ proved.

(iv) LHS $= \sin 2A + \sin 2B - \sin 2C$

$= \sin 2A - \sin 2C + \sin 2B$

$= (\sin 2A - \sin 2C) + \sin 2B$

$= 2\cos(A+C)\sin(A-C) + 2\sin B \cos B$

$= -2\cos B \sin(A-C) + 2\sin B \cos B$ $\quad [\cos(A+C) = \cos(\pi - B) = -2\cos B]$

$= 2\cos B[\sin B - \sin(A-C)]$

$= 2\cos B[\sin(A+C) - \sin(A-C)]$ $\quad [\sin B = \sin[\pi - (A+C)] = \sin(A+C)]$

$= 2\cos B[2\cos A \sin C] = 4\cos A \cos B \sin C = \text{RHS}$ proved.

(v) LHS $= \sin A + \sin B + \sin C$

$= 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2} + 2\sin\dfrac{C}{2}\cos\dfrac{C}{2}$

$= 2\cos\dfrac{C}{2}\cos\dfrac{A-B}{2} + 2\sin\dfrac{C}{2}\cos\dfrac{C}{2}$ $\quad [\sin\dfrac{A+B}{2} = \sin(\dfrac{\pi}{2} - \dfrac{C}{2}) = \cos\dfrac{C}{2}]$

$= 2\cos\dfrac{C}{2}(\cos\dfrac{A-B}{2} + \sin\dfrac{C}{2})$

$= 2\cos\dfrac{C}{2}(\cos\dfrac{A-B}{2} + \cos\dfrac{A+B}{2})$ $\quad [\cos\dfrac{A+B}{2} = \cos(\dfrac{\pi}{2} - \dfrac{C}{2}) = \sin\dfrac{C}{2}]$

$= 2\cos\dfrac{C}{2}(2\cos\dfrac{A}{2}\cos\dfrac{B}{2}) = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \text{RHS}$ proved.

(vi) LHS $= \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + (2\cos^2 C - 1)$

$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$ $\quad [\cos(A+B) = \cos(\pi - C) = -\cos C]$

$= -2\cos C(\cos(A-B) - \cos C) - 1$

$= -2\cos C[\cos(A-B) + \cos(A+B)] - 1$ $\quad [-\cos C = \cos(A+B)]$

$= -2\cos C\, 2\cos A \cos B - 1$

$= -4\cos A \cos B \cos C - 1 = \text{RHS}$ proved.

(vii) LHS $= \cos A + \cos B + \cos C$

$= 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2} + \cos C$

$= 2\sin\dfrac{C}{2}\cos\dfrac{A-B}{2} + 1 - 2\sin^2\dfrac{C}{2}$ $\quad [\cos\dfrac{A+B}{2} = \cos(\dfrac{\pi}{2} - \dfrac{C}{2}) = \sin\dfrac{C}{2}]$

$= 2\sin\dfrac{C}{2}[\cos\dfrac{A-B}{2} - \sin\dfrac{C}{2}] + 1$

$= 2\sin\dfrac{C}{2}[\cos\dfrac{A-B}{2} - \cos\dfrac{A+B}{2}] + 1$

$= 1 + 2\sin\dfrac{C}{2}\, 2\sin\dfrac{A}{2}\sin\dfrac{B}{2} = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \text{RHS}$ proved.

(viii) LHS $= \sin^2 A + \sin^2 B + \sin^2 C$

$= \dfrac{1}{2}(2\sin^2 A + 2\sin^2 B + 2\sin^2 C)$

$= \dfrac{1}{2}(1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C)$

$= \dfrac{3}{2} - \dfrac{1}{2}(\cos 2A + \cos 2B + \cos 2C)$

$= \dfrac{3}{2} - \dfrac{1}{2}(-4\cos A \cos B \cos C - 1)$ see Ex 1(vi)

$= 2 + 2\cos A \cos B \cos C = \text{RHS}$ proved.

(iii) $\tan A+\cot A=2\operatorname{cosec}2A$ (iv) $\cos^6\theta+\sin^6\theta=\frac{1}{4}(1+3\cos^2 2\theta)$

(v) $\cot A=\frac{1}{2}(\cot\frac{A}{2}-\tan\frac{A}{2})$ (vi) $\frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}=2$

(vii) $\frac{1+\sin x}{1-\sin x}=\tan^2(\frac{\pi}{4}+\frac{x}{2})$ (viii) $\tan(\frac{\pi}{4}+\theta)=\frac{1+\sin 2\theta}{\cos 2\theta}$

(ix) $\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}=\frac{1+\cos\theta}{\sin\theta}$ (x) $\tan(\frac{\pi}{4}+\theta)+\tan(\frac{\pi}{4}-\theta)=2\sec 2\theta$

(xi) $\frac{\cos A-\sin A}{\cos A+\sin A}=\sec 2A-\tan 2A$ (xii) $\frac{\sin 4\theta(1-\cos 2\theta)}{\cos 2\theta(1-\cos 4\theta)}=\tan\theta$

7. If $\tan\frac{A}{2}=t$, then prove that $\sin A+\tan A=\frac{4t}{1-t^4}$

* * *

2.5 Trigonometric Identities

Many interesting identities can be established for the relation amongst three angles of a triangle. To establish these identities, some necessary properties of supplementary and complimentary angles are required.

If $A+B+C=\pi$

then $B+C=\pi-A$ $(\pi=180^\circ)$

$\therefore \sin(B+C)=\sin(\pi-A)=\sin A$

$\therefore \cos(B+C)=\cos(\pi-A)=-\cos A$

$\therefore \tan(B+C)=\tan(\pi-A)=-\tan A$

Similar result can be obtained for $A+C=\pi-B$ and $A+B=\pi-C$.

Again $A+B+C=\pi$ then $B+C=\pi-A$

$\therefore \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$

Worked out examples :

Ex 1. If $A+B+C=\pi$ then prove that

(i) **[QP 2013]** $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(ii) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

(iii) **[QP 2009, 2015, 2017]** $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(iv) **[QP 2011, 2014]** $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$

(v) **[QP 2014, 2015]** $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(vi) $\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1$

(vii) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(viii) $\sin^2A+\sin^2B+\sin^2C=2+2\cos A\cos B\cos C$

(ix) $\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C=1$

(x) $\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(xi) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$

(xii) $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$

(xiii) $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin A\sin C}+\frac{\cos C}{\sin A\sin B}=2$

(xiv) $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$

(xv) $\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$

Soln: (i) Given $A+B+C=\pi$

$\Rightarrow A+B=\pi-C$

$\Rightarrow \tan(A+B)=\tan(\pi-C)$

$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

$\Rightarrow \tan A+\tan B=-\tan C(1-\tan A\tan B)$

$\Rightarrow \tan A+\tan B=-\tan C+\tan A\tan B\tan C$

$\Rightarrow \tan A+\tan B+\tan C=\tan A\tan B\tan C$ proved.

(ii) Given $A+B+C=\pi$ $\Rightarrow A+B=\pi-C$

$$\therefore \tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$$ proved

(iv) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$

$$=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}\frac{\frac{5}{6}}{\frac{5}{6}}=\tan^{-1}1=\frac{\pi}{4}$$ proved

2.9.3 Corollary: (i) If $x=y$, then $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$

(ii) If $x=y=z$, then $3\tan^{-1}x=\tan^{-1}\frac{3x-x^3}{1-3x^2}$

Ex 3. Prove that (i) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}$

(ii) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{(1-x^2)(1-y^2)}\}$

Proof: (i) Let $\theta=\sin^{-1}x$ $\therefore \sin\theta=x \Rightarrow \cos\theta=\sqrt{1-x^2}$

and $\alpha=\sin^{-1}y \Rightarrow \sin\alpha=y \Rightarrow \cos\alpha=\sqrt{1-y^2}$

Now, $\sin(\theta\pm\alpha)=\sin\alpha\cos\theta\pm\cos\alpha\sin\theta$

$=y\sqrt{1-x^2}\pm x\sqrt{1-y^2}$

$\Rightarrow \theta\pm\alpha=\sin^{-1}(y\sqrt{1-x^2}\pm x\sqrt{1-y^2})$

$\Rightarrow \sin^{-1}x\pm\sin^{-1}y=\sin^{-1}(y\sqrt{1-x^2}\pm x\sqrt{1-y^2})$ proved

(ii) Let $\theta=\cos^{-1}x$ $\therefore \cos\theta=x \Rightarrow \sin\theta=\sqrt{1-x^2}$

and $\alpha=\cos^{-1}y \Rightarrow \cos\alpha=y \Rightarrow \sin\alpha=\sqrt{1-y^2}$

Now, $\cos(\theta\pm\alpha)=\cos\theta\cos\alpha\mp\sin\theta\sin\alpha$

$=xy\mp\sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \theta\pm\alpha=\cos^{-1}(xy\mp\sqrt{1-x^2}\sqrt{1-y^2})$

$\Rightarrow \cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{(1-x^2)(1-y^2)}\}$ proved

2.9.4 Corollary: (i) If $x=y$, then $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

(ii) If $x=y$, then $2\cos^{-1}x=\cos^{-1}\{x^2-(1-x^2)\}=\cos^{-1}\{2x^2-1\}$

Ex 4. Prove that (i) $\sin^{-1}(-x)=-\sin^{-1}x$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x$

Soln: (i) Let $\sin^{-1}(-x)=\theta$

$\Rightarrow -x=\sin\theta \Rightarrow x=-\sin\theta=\sin(-\theta)$

$\Rightarrow \sin^{-1}x=-\theta \Rightarrow \theta=-\sin^{-1}x$

$\Rightarrow \sin^{-1}(-x)=-\sin^{-1}x$ proved.

(ii) Let $\cos^{-1}(-x)=\theta$

$\Rightarrow -x=\cos\theta$

$x=-\cos\theta=\cos(\pi-\theta)$

$\Rightarrow \cos^{-1}x=\pi-\theta$

$\Rightarrow \theta=\pi-\cos^{-1}x$

$\Rightarrow \cos^{-1}(-x)=\pi-\cos^{-1}x$ proved.

(iii) Let $\tan^{-1}(-x)=\theta$

$\Rightarrow -x=\tan\theta \Rightarrow x=-\tan\theta=\tan(-\theta)$

$\Rightarrow \tan^{-1}x=-\theta \Rightarrow \theta=-\tan^{-1}x$

$\Rightarrow \tan^{-1}(-x)=-\tan^{-1}x$ proved.

Ex 5. Prove that (i) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$ (ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(iii) $3\tan^{-1}x=\tan^{-1}\frac{3x-x^3}{1-3x^2}$

Proof: (i) Let $\theta=\sin^{-1}x$ $\therefore \sin\theta=x$

Now $\sin 3=3\sin\theta-4\sin^3\theta$

$\Rightarrow \sin 3\theta=3x-4x^3$

$\Rightarrow 3\theta=\sin^{-1}(3x-4x^3)$

In a similar way, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}}$ and $\cot\theta = \dfrac{\sqrt{1-x^2}}{x}$

$$\therefore \theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$\therefore \theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \operatorname{cosec}^{-1}\frac{1}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

Worked out examples :

Ex 1. Prove that (i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ (ii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2}$

(iii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$ (iv)**[Q.P 2010]** $\sin^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$

Proof: (i) Let $\theta = \sin^{-1}x$ $\quad \therefore \sin\theta = x = \cos\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore \dfrac{\pi}{2} - \theta = \cos^{-1}x \Rightarrow \dfrac{\pi}{2} = \theta + \cos^{-1}x = \sin^{-1}x + \cos^{-1}x$ proved

(ii) Let $\theta = \sec^{-1}x$ $\quad \therefore \sec\theta = x = \operatorname{cosec}\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore \dfrac{\pi}{2} - \theta = \operatorname{cosec}^{-1}x \Rightarrow \dfrac{\pi}{2} = \theta + \operatorname{cosec}^{-1}x = \sec^{-1}x + \operatorname{cosec}^{-1}x$ proved

(iii) Let $\theta = \tan^{-1}x$ $\quad \therefore \tan\theta = x = \cot\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore \dfrac{\pi}{2} - \theta = \cot^{-1}x \Rightarrow \dfrac{\pi}{2} = \theta + \cot^{-1}x = \tan^{-1}x + \cot^{-1}x$ proved

(iv) $\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$, $\cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{3}$ $\Rightarrow \sin^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{6} + \dfrac{\pi}{3} = \dfrac{\pi}{2}$ proved

Ex 2. Prove that (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$

(ii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$ (iv)**[Q.P 2013]** $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$

Proof: (i) Let $\tan^{-1}x = \alpha$ and $\tan^{-1}y = \beta$

$\therefore x = \tan\alpha$ and $y = \tan\beta$

Now, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\therefore \tan(\alpha+\beta) = \dfrac{x+y}{1-xy} \Rightarrow (\alpha+\beta) = \tan^{-1}\dfrac{x+y}{1-xy}$

$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$ proved

(ii) Let $\tan^{-1}x = \alpha$; $\tan^{-1}y = \beta$; and $\tan^{-1}z = \gamma$;

$\Rightarrow x = \tan\alpha$; $\Rightarrow y = \tan\beta$; and $\Rightarrow z = \tan\gamma$;

Now, $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\gamma\tan\alpha - \tan\beta\tan\gamma}$

$\therefore \alpha + \beta + \gamma = \tan^{-1}\dfrac{x+y+z-xyz}{1-xy-zx-yz}$

$\Rightarrow \tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}$ proved

(iii) Let $\tan^{-1}x = \alpha$ and $\tan^{-1}y = \beta$

$\therefore x = \tan\alpha$ and $y = \tan\beta$

Now, $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\therefore \tan(\alpha-\beta) = \dfrac{x-y}{1+xy} \Rightarrow (\alpha-\beta) = \tan^{-1}\dfrac{x-y}{1+xy}$

$= \frac{1}{2}[2\cos(B+C)\cos(B-C) - 2\cos A\cos(B+C)] + 1$

$[\cos(B+C) = \cos(-A) = \cos A]$

$= \cos A\,[\cos(B-C) - \cos(B+C)] + 1$

$= 1 + 2\cos A \sin B \sin C = \text{RHS}$ proved

(v) $56^\circ + 40^\circ + 84^\circ = 180^\circ$

Hence by result $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ we have

$\sin 56^\circ + \sin 40^\circ + \sin 84^\circ = 4\cos 28^\circ\cos 20^\circ\cos 42^\circ$

Exercise 2.5

1. If $A+B+C = \pi$ then prove that

(i) $\sin^2 A + \sin^2 B - \cos^2 C = 1 + 2\cos A\cos B\cos C$

(ii) $\cos^2 A + \cos^2 B - \sin^2 C + 2\cos A\cos B\cos C = 0$

(iii) $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C$

(iv) $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

(v) $\sin(B-A+C) + \sin(C-B+A) + \sin(A-C+B) = 4\sin A\sin B\sin C$

(vi) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(vii) $\tan 2A + \tan 2B + \tan 2C = \tan 2A\tan 2B\tan 2C$

(viii) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(ix) $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(x) $\cos^2 2A + \cos^2 2B - \sin^2 2C = 1 + 2\cos 2A\cos 2B\cos 2C$

2. If $A+B+C = 0$ then prove that

(i) $\cos A + \cos B + \cos C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 1$

(ii) $\cos A\cos B\cos C\,(\tan A + \tan B + \tan C) = \sin A\sin B\sin C$

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2.6 Inverse circular functions

In the equation $\sin\theta = x$, θ is such an angle whose sine ratio is x. Again if $\sin\theta = x$, then θ can be written as $\sin^{-1}x$. The symbol $\sin^{-1}x$ is usually read as 'sine inverse x'.

Note: 1. The subscript -1 is used here as an inverse notation and not an index.

2. $\sin^{-1}x \neq (\sin x)^{-1}$

3. $\sin^{-1}x$ is an angle, where x is a number. $\sin\theta$ is a number, where θ is an angle.

2.6.1 General and principle values:

We know that there are infinitely many angle of which sine ratio is x.

Thus if $\sin\theta = \frac{1}{2}$, then $\sin^{-1}(\frac{1}{2}) = \theta = n\pi + (-1)^n\frac{\pi}{6}$

The smallest numerical value either positive or negative of θ is called the *principal value* and we adopt that value for $\sin^{-1}x$. Accordingly $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$. If corresponding to the same ratio there are two numerically equal values, one positive and the other negative, it is customary to take the positive angle as the principal value. In a similar way we define $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$. These expressions are called *Inverse Circular Functions*. Thus we see

$\sin^{-1}(-\frac{1}{2}) = -30^\circ$ $\cos^{-1}(\frac{1}{2}) = 60^\circ$ $\tan^{-1}1 = 45^\circ$

2.6.2 Relation between the inverse circular functions:

Let $\sin^{-1}x = \theta$ $\Rightarrow \sin\theta = x$;

$\Rightarrow \operatorname{cosec}\theta = \frac{1}{x}$; $\Rightarrow \theta = \operatorname{cosec}^{-1}\frac{1}{x}$

Now, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}$;

$\therefore \theta = \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$

a. $\sin^2 A + \sin^2 B + \sin^2 C = 2 - 2\cos A\cos B\cos C$

b. $\sin(A+B-C)+\sin(B+C-A)+\sin(C+A-B)$
$= 2\cos A\cos B\sin C + 2\sin A\sin B\sin C$

(iii) If $A+B+C = 360°$ then prove that
$1-\cos^2 A - \cos^2 B - \cos^2 C + 2\cos A\cos B\cos C = 0$

(iv) If $A+B+C=0$ then prove that
$\cos^2 C + \cos^2 B - \cos^2 A = 1 + 2\cos A\sin B\sin C$

(v) Prove- $\sin 56° + \sin 40° + \sin 84° = 4\cos 28°\cos 20°\cos 42°$

Soln: (i) a. RHS $= \tan B + \tan C = \dfrac{\sin B}{\cos B} + \dfrac{\sin C}{\cos C}$

$= \dfrac{\sin B\cos C + \cos B\sin C}{\cos B\cos C} = \dfrac{\sin(B+C)}{\cos A}$ given $\cos A = \cos B\cos C$

$= \dfrac{\sin A}{\cos A} = \tan A = \text{LHS}$ [Given $A+B+C=\pi \Rightarrow B+C=\pi - A$]

b. LHS $= \cot B\cot C = \dfrac{\cos B\cos C}{\sin B\sin C} = \dfrac{\cos A}{\sin B\sin C}$ Given condition

$= \dfrac{-\cos(B+C)}{\sin B\sin C} = -\left(\dfrac{\cos B\cos C}{\sin B\sin C} - \dfrac{\sin B\sin C}{\sin B\sin C}\right)$

$= -\cot B\cot C + 1$

$\Rightarrow 2\cot B\cot C = 1 \Rightarrow \cot B\cot C = \dfrac{1}{2} = \text{RHS}$ proved.

(ii) a. LHS $= \sin^2 A + \sin^2 B + \sin^2 C$

$= \dfrac{1}{2}(2\sin^2 A + 2\sin^2 B + 2\sin^2 C)$

$= \dfrac{1}{2}(1-\cos 2A + 1-\cos 2B + 1-\cos 2C)$

$= \dfrac{3}{2} - \dfrac{1}{2}(\cos 2A + \cos 2B + \cos 2C)$

$= \dfrac{3}{2} - \dfrac{1}{2}[2\cos(A+B)\cos(A-B) + (2\cos^2 C - 1)]$

$= \dfrac{3}{2} - \dfrac{1}{2}[2\cos C\cos(A-B) + 2\cos^2 C - 1]$ [$\cos(A+B) = \cos C$]

$= \dfrac{3}{2} - \dfrac{1}{2}[2\cos C(\cos(A-B) + \cos C) - 1]$

$= \dfrac{3}{2} - \dfrac{1}{2}[2\cos C\{\cos(A-B) + \cos(A+B)\} - 1]$ [$\cos C = \cos(A+B)$]

$= \dfrac{3}{2} + \dfrac{1}{2} - \dfrac{1}{2}[2\cos C\, 2\cos A\cos B] = 2 - 2\cos A\cos B\cos C = \text{RHS}$

b. LHS $= \sin(A+B-C) + \sin(B+C-A) + \sin(C+A-B)$
$= \sin 0 + \sin(B+A+B-A) + \sin(A+B+A-B)$
$= \sin 2B + \sin 2A$
$= 2\sin(A+B)\cos(A-B)$
$= 2\sin C(\cos A\cos B + \sin A\sin B)$
$= 2\cos A\cos B\sin C + 2\sin A\sin B\sin C = \text{RHS}$ proved

(iii) Let us compute $\cos^2 A + \cos^2 B + \cos^2 C$
$\cos^2 A + \cos^2 B + \cos^2 C$

$= \dfrac{1}{2}(2\cos^2 A + 2\cos^2 B + 2\cos^2 C)$

$= \dfrac{1}{2}(\cos 2A + \cos 2B + 2\cos^2 C + 2)$

$= \dfrac{1}{2}[2\cos(A+B)\cos(A-B) + 2\cos C\cos(A+B)] + 1$

[$\cos(A+B) = \cos(2\pi - C) = \cos C$]

$= \cos C[\cos(A-B) + \cos(A+B)] + 1$
$= 2\cos A\cos B\cos C + 1$

Now LHS $= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2\cos A\cos B\cos C$
$= 1 - (2\cos A\cos B\cos C + 1) + 2\cos A\cos B\cos C = 0 = \text{RHS}$ proved

(iv) LHS $= \cos^2 C + \cos^2 B - \cos^2 A$

$= \dfrac{1}{2}(2\cos^2 C + 2\cos^2 B - 2\cos^2 A)$

$= \dfrac{1}{2}(\cos 2B + \cos 2C - 2\cos^2 A + 2)$

$$\Rightarrow \theta = \cos^{-1}x = 2\cos^{-1}\sqrt{\frac{1+x}{2}} \qquad (2)$$

$$\cos\theta = x \quad \Rightarrow \frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}$$

$$\Rightarrow \theta = \cos^{-1}x = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} \qquad (3)$$

From (1), (2) and (3) we have

$$\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Ex 15. Solve: (i) $\cot^{-1}x + \tan^{-1}3 = \frac{\pi}{2}$ (ii) $\tan^{-1}x = \cot^{-1}x$

(iii) $\tan(\cos^{-1}x) = \sin(\cot^{-1}\frac{1}{2})$ (iv) $\cot^{-1}x + \cot^{-1}2x = \frac{3\pi}{4}$

(v) $\cos^{-1}\frac{5}{x} + \cos^{-1}\frac{12}{x} = \frac{\pi}{2}$

Soln: (i) $\cot^{-1}x + \tan^{-1}3 = \tan^{-1}\frac{1}{x} + \tan^{-1}3 = \tan^{-1}\frac{\frac{1}{x}+3}{1-\frac{3}{x}} = \tan^{-1}\frac{x+3}{x-3}$

$\cot^{-1}x + \tan^{-1}3 = \frac{\pi}{2} \Rightarrow \tan^{-1}\frac{x+3}{x-3} = \frac{\pi}{2} \Rightarrow x-3=0$

$\Rightarrow x = 3$

Solved

(ii) $\tan^{-1}x = \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x \Rightarrow 2\tan^{-1}x = \frac{\pi}{2} \Rightarrow x = \tan\frac{\pi}{4} = 1$

$\Rightarrow x = 1$

Solved

(iii) $\tan(\cos^{-1}x) = \tan\sec^{-1}\frac{1}{x} = \tan\tan^{-1}\sqrt{\frac{1}{x^2}-1} = \frac{\sqrt{1-x^2}}{x}$

$\sin(\cot^{-1}\frac{1}{2}) = \sin\operatorname{cosec}^{-1}\sqrt{\frac{1}{2^2}+1} = \sin\sin^{-1}\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}}$

$\tan(\cos^{-1}x) = \sin(\cot^{-1}\frac{1}{2}) \Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow 5(1-x^2) = 4x^2$

$\Rightarrow 5(1-x^2) = 4x^2 \Rightarrow 9x^2 = 5 \Rightarrow x = \pm\frac{\sqrt{5}}{3}$ Solved

(iv) $\cot^{-1}x + \cot^{-1}2x = \cot^{-1}\frac{x\times 2x-1}{2x+x} = \cot^{-1}\frac{2x^2-1}{3x}$

$\cot^{-1}x + \cot^{-1}2x = \frac{3\pi}{4} \Rightarrow \cot^{-1}\frac{2x^2-1}{3x} = \frac{3\pi}{4} = \cot^{-1}(-1)$

$\Rightarrow 2x^2 + 3x - 1 = 0 \Rightarrow x = \frac{-3\pm\sqrt{17}}{4}$ Solved

(v) $\cos^{-1}\frac{5}{x} + \cos^{-1}\frac{12}{x} = \cos^{-1}\{\frac{5}{x}\times\frac{12}{x} - \sqrt{(1-\frac{25}{x^2})(1-\frac{144}{x^2})}\}$

$= \cos^{-1}\{\frac{60}{x^2} - \frac{\sqrt{(x^2-25)(x^2-144)}}{x^2}\}$

Given $\cos^{-1}\frac{5}{x} + \cos^{-1}\frac{12}{x} = \frac{\pi}{2}$

$\Rightarrow \frac{60}{x^2} - \frac{\sqrt{(x^2-25)(x^2-144)}}{x^2} = 0$

$\Rightarrow 3600 = (x^2-25)(x^2-144) \Rightarrow x^4 - 169x^2 = 0 \Rightarrow x = 13$ Solved

(ix) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \tan^{-1}1$

(x) $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Proof: (i) LHS $= \sin(\cos^{-1}x) = \sin(\sin^{-1}\sqrt{1-x^2}) = \sqrt{1-x^2}$

$= \cos(\cos^{-1}\sqrt{1-x^2}) = \cos(\sin^{-1}x) =$ RHS Proved

(ii) LHS $= \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} = \cos^{-1}\frac{1}{\sqrt{1+(\frac{x-y}{1+xy})^2}}$

$= \cos^{-1}\frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}} =$ RHS Proved

(iii) LHS $= \tan^{-1}x + \cot^{-1}y = \tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}\frac{x+\frac{1}{y}}{1-\frac{x}{y}} = \tan^{-1}\frac{xy+1}{y-x} =$ RHS

(iv) LHS $= \tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}x + \tan^{-1}\frac{1}{x+1} = \tan^{-1}\frac{x+\frac{1}{x+1}}{1-\frac{x}{x+1}}$

$= \tan^{-1}(1+x+x^2) =$ RHS Proved

(v) LHS $= \tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z)$

$= \tan\{(\frac{\pi}{2} - \cot^{-1}x) + (\frac{\pi}{2} - \cot^{-1}y) + (\frac{\pi}{2} - \cot^{-1}z)\}$

$= \tan(\pi + \frac{\pi}{2} - (\cot^{-1}x + \cot^{-1}y + \cot^{-1}z))$

$= \tan(\frac{\pi}{2} - (\cot^{-1}x + \cot^{-1}y + \cot^{-1}z))$

$= \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z) =$ RHS Proved

(vi) LHS $= \sin\cos^{-1}\tan\sec^{-1}x = \sin\cos^{-1}\tan\tan^{-1}\sqrt{x^2-1}$

$= \sin\cos^{-1}\sqrt{x^2-1} = \sin\{\sin^{-1}\sqrt{(1-(\sqrt{x^2-1})^2)}\}$

$= \sin\sin^{-1}\sqrt{2-x^2} = \sqrt{2-x^2} =$ RHS Proved

(vii) LHS $= \tan^{-1}\frac{1}{a+b} + \tan^{-1}\frac{b}{a^2+ab+1} = \tan^{-1}\frac{1}{a}$

$= \tan^{-1}\frac{\frac{1}{a+b} + \frac{b}{a^2+ab+1}}{1 - \frac{1}{a+b} \times \frac{b}{a^2+ab+1}} = \tan^{-1}\frac{a^2+ab+1+ab+b^2}{a(a^2+ab+1+ab+b^2)+b-b}$

$= \tan^{-1}\frac{1}{a} =$ RHS Proved

(viii) LHS $= \tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1}\frac{c-d}{1+cd}$

$= (\tan^{-1}a - \tan^{-1}b) + (\tan^{-1}b - \tan^{-1}c) + (\tan^{-1}c - \tan^{-1}d)$

$= \tan^{-1}a - \tan^{-1}d =$ RHS Proved

(ix) LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\times\frac{3}{5}} = \tan^{-1}1 =$ RHS Proved

(x) $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}} = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let $\cos^{-1}x = \theta \quad \Rightarrow \cos\theta = x \quad \Rightarrow 1-x = 1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$\Rightarrow \theta = \cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$ (1)

$\cos\theta = x \quad \Rightarrow 1+x = 1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$\tan 2\theta = \frac{2x}{1-x^2} \Rightarrow 2\theta = 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} \quad (3)$$

Thus from (1), (2) and (3) we have

$$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2} \quad \text{Proved}$$

Ex 9. [QP 2011] Prove that $\tan^{-1}x = \sec^{-1}\sqrt{1+x^2}$

Proof: Let $\sec^{-1}\sqrt{1+x^2} = z \qquad \therefore \sqrt{1+x^2} = \sec z$

We have $1+\tan^2 z = \sec^2 z$

$\Rightarrow 1+\tan^2 z = 1+x^2$

$\Rightarrow \tan^2 z = x^2 \Rightarrow \tan z = x \Rightarrow z = \tan^{-1}x$

$\Rightarrow \tan^{-1}x = \sec^{-1}\sqrt{1+x^2}$ Proved.

Ex 10. Prove that $2\tan^{-1}\frac{1+x}{1-x} + \sin^{-1}\frac{1-x^2}{1+x^2} = \pi$

Proof: Let $x = \tan\theta$

Now, $2\tan^{-1}\frac{1+x}{1-x} = 2\tan^{-1}\frac{1+\tan\theta}{1-\tan\theta} = 2\tan^{-1}[\tan(\frac{\pi}{4}+\theta)] = 2(\frac{\pi}{4}+\theta) = \frac{\pi}{2}+2\theta$

Again, $\sin^{-1}\frac{1-x^2}{1+x^2} = \sin^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta} = \sin^{-1}\cos(2\theta) = \sin^{-1}[\sin(\frac{\pi}{2}-2\theta)] = \frac{\pi}{2}-2\theta$

Now, $2\tan^{-1}\frac{1+x}{1-x} + \sin^{-1}\frac{1-x^2}{1+x^2} = \frac{\pi}{2}+2\theta+\frac{\pi}{2}-2\theta = \pi$ proved

Ex 11. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z = \pi$ prove that

$$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2} = 2xyz$$

Soln: Let $\sin^{-1}x = A \Rightarrow x = \sin A \Rightarrow \cos A = \sqrt{1-x^2}$

$\sin^{-1}y = B \Rightarrow y = \sin B \Rightarrow \cos B = \sqrt{1-y^2}$

$\sin^{-1}z = C \Rightarrow z = \sin C \Rightarrow \cos C = \sqrt{1-z^2}$

Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z = \pi \Rightarrow A+B+C = \pi$

$\therefore \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

$\Rightarrow 2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C = 4\sin A\sin B\sin C$

$\Rightarrow \sin A\cos A + \sin B\cos B + \sin C\cos C = 2\sin A\sin B\sin C$

$\Rightarrow x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2} = 2xyz$ proved

Ex 12. [QP 2013] Find the value of $\sin(\sin^{-1}\frac{12}{13}+\sin^{-1}\frac{4}{5})$

Soln: $\sin(\sin^{-1}\frac{12}{13}+\sin^{-1}\frac{4}{5}) = \sin(\sin^{-1}(\frac{4}{5}\sqrt{1-\left(\frac{12}{13}\right)^2}+\frac{12}{13}\sqrt{1-\left(\frac{4}{5}\right)^2})$

$[\sin^{-1}x+\sin^{-1}y = \sin^{-1}(y\sqrt{1-x^2}+x\sqrt{1-y^2})]$

$= \sin(\sin^{-1}(\frac{4}{5}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{3}{5}) = \sin\sin^{-1}(\frac{56}{65}) = \frac{56}{65}$ Proved

Ex 13. Prove that $\cos(2\sin^{-1}x) = 1-2x^2$

Proof: Let $\theta = \sin^{-1}x \quad \therefore \sin\theta = x$

Now, $\cos 2\theta = 1-2\sin^2\theta = 1-2x^2$

$\Rightarrow \cos(2\sin^{-1}x) = 1-2x^2$ Proved

Ex 14. Prove that

(i) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x)$

(ii) $\tan^{-1}x - \tan^{-1}y = \cos^{-1}\frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}}$

(iii) $\tan^{-1}x + \cot^{-1}y = \tan^{-1}\frac{xy+1}{y-x}$

(iv) $\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(1+x+x^2)$

(v) $\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z) = \cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z)$

(vi) $\sin\cos^{-1}\tan\sec^{-1}x = \sqrt{2-x^2}$

(vii) $\tan^{-1}\frac{1}{a+b}+\tan^{-1}\frac{b}{a^2+ab+1} = \tan^{-1}\frac{1}{a}$

(viii) $\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}\frac{c-d}{1+cd} = \tan^{-1}a-\tan^{-1}d$

$\Rightarrow 3\sin^{-1}x = \sin^{-1}(3x-4x^3)$ proved

(ii) Let $\theta = \cos^{-1}x$ $\therefore \cos\theta = x$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos 3\theta = 4x^3 - 3x$

$\Rightarrow 3\theta = \cos^{-1}(4x^3-3x)$

$\Rightarrow 3\cos^{-1}x = \cos^{-1}(4x^3-3x)$ proved

(iii) Let $\theta = \tan^{-1}x$ $\therefore \tan\theta = x$

Now $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

$\Rightarrow \tan 3\theta = \dfrac{3x-x^3}{1-3x^2} \Rightarrow 3\theta = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

$\Rightarrow 3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$ proved

Ex 6. Prove that $\tan^{-1}\dfrac{5}{12} = \sin^{-1}\dfrac{5}{13} = \cos^{-1}\dfrac{12}{13}$

Proof: Let $\theta = \tan^{-1}\dfrac{5}{12}$ $\therefore \tan\theta = \dfrac{5}{12} \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \dfrac{5}{12}$

$\Rightarrow \dfrac{\sin\theta}{5} = \dfrac{\cos\theta}{12} = \dfrac{\sqrt{\sin^2\theta + \cos^2\theta}}{\sqrt{5^2+12^2}} = \dfrac{1}{13}$

$\therefore \sin\theta = \dfrac{5}{13}$ and $\cos\theta = \dfrac{12}{13}$

$\therefore \theta = \tan^{-1}\dfrac{5}{12} = \sin^{-1}\dfrac{5}{13} = \cos^{-1}\dfrac{12}{13}$ proved

Ex 7. [Q.P 2013] If $\sin(A-B) = \dfrac{1}{2}$ and $\cos(A+B) = \dfrac{1}{2}$ then find A and B.

Soln: Given $\sin(A-B) = \dfrac{1}{2}$ $\therefore A - B = \sin^{-1}\dfrac{1}{2} = \sin^{-1}(\sin 30^\circ) = 30^\circ$

$\therefore A - B = 30^\circ$........(1)

Again $\cos(A+B) = \dfrac{1}{2} \Rightarrow A+B = \cos^{-1}\dfrac{1}{2} = \cos^{-1}(\cos 60^\circ) = 60^\circ$

$\Rightarrow A+B = 60^\circ$.................(2)

Now, (1) + (2) $\Rightarrow 2A = 30^\circ + 60^\circ \Rightarrow 2A = 90^\circ \Rightarrow A = 45^\circ$

Again, (2) – (1) $\Rightarrow 2B = 60^\circ - 30^\circ \Rightarrow 2B = 30^\circ \Rightarrow B = 15^\circ$

$\therefore A = 45^\circ$ and $B = 15^\circ$

Ex 8. Prove that (i) $\tan^{-1}\dfrac{x-y}{1+xy} + \tan^{-1}\dfrac{y-z}{1+yz} + \tan^{-1}\dfrac{z-x}{1+zx} = 0$

(ii) $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2} = \tan^{-1}\dfrac{2x}{1-x^2}$

Proof: (i) We know, $\tan^{-1}\dfrac{x-y}{1+xy} = \tan^{-1}x - \tan^{-1}y$

Similarly, $\tan^{-1}\dfrac{y-z}{1+yz} = \tan^{-1}y - \tan^{-1}z$;

and $\tan^{-1}\dfrac{z-x}{1+zx} = \tan^{-1}z - \tan^{-1}x$

Now, LHS $= \tan^{-1}\dfrac{x-y}{1+xy} + \tan^{-1}\dfrac{y-z}{1+yz} + \tan^{-1}\dfrac{z-x}{1+zx} = 0$

$= \tan^{-1}x - \tan^{-1}y + \tan^{-1}y - \tan^{-1}z + \tan^{-1}z - \tan^{-1}x$

$= 0 =$ R.H.S. Proved

(ii) Let $\theta = \tan^{-1}x$ $\therefore \tan\theta = x$

We already have

$\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$, $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

This means $\sin 2\theta = \dfrac{2x}{1+x^2} \Rightarrow 2\theta = 2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2}$ (1)

$\cos 2\theta = \dfrac{1-x^2}{1+x^2} \Rightarrow 2\theta = 2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1+x^2}$ (2)

$$= \frac{(a+b)^2 - c^2}{2ab} = \frac{(a+b+c)(a+b-c)}{2ab}$$

$$= \frac{(a+b+c)(a+b+c-2c)}{2ab} = \frac{2s(2s-2c)}{2ab}$$

$$\Rightarrow \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

(v) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$

$$= 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc} = \frac{2\Delta}{bc}$$

since $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

(vi) **[QP 2013]** $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

$$\text{RHS} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}\cot\frac{A}{2}$$

$$= \frac{\sin B - \sin C}{\sin B + \sin C}\cot\frac{A}{2}$$

$$= \frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}\cot\frac{A}{2}$$

$$= \tan\frac{B-C}{2}\cot\frac{B+C}{2}\cot\frac{A}{2}$$

$$= \tan\frac{B-C}{2}\tan\frac{A}{2}\cot\frac{A}{2} = \tan\frac{B-C}{2} = \text{LHS}$$ Proved

$$[\therefore \cot\frac{B+C}{2} = \cot(90^\circ - \frac{A}{2}) = \tan\frac{A}{2}]$$

Ex 2. [QP 2009, 2013, 2015] Prove that $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

Proof: We have, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$

$\therefore a\sin(B-C) = 2R\sin A\sin(B-C) = 2R\sin(B+C)\sin(B-C)$

$= 2R(\sin^2 B - \sin^2 C)$

Now, $b\sin(C-A) = 2R(\sin^2 C - \sin^2 A)$

And, $c\sin(A-B) = 2R(\sin^2 A - \sin^2 B)$

LHS $= a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= 2R(\sin^2 B - \sin^2 C) + 2R(\sin^2 C - \sin^2 A) + 2R(\sin^2 A - \sin^2 B)$

$= 0 =$ RHS Proved.

Ex 3. [QP 2009, 2010, 2013] Prove that

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$$

Proof: We have, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$

$a(\sin B - \sin C) = 2R\sin A(\sin B - \sin C) = 2R\sin A\sin B - 2R\sin A\sin C$ (1)

Similarly $b(\sin C - \sin A) = 2R\sin B(\sin C - \sin A) = 2R\sin B\sin C - 2R\sin B\sin A$ (2)

And, $c(\sin A - \sin B) = 2R\sin C(\sin A - \sin B) = 2R\sin C\sin A - 2R\sin C\sin B$ (3)

From (1), (2) and (3) we have

LHS $= a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0 =$ RHS proved.

Ex 4. In a triangle ABC, prove that $(a^2 - b^2 - c^2)\tan A + (c^2 + a^2 - b^2)\tan B = 0$.

Proof: We have, $a^2 = b^2 + c^2 - 2bc\cos A$ as well as $b^2 = c^2 + a^2 - 2ca\cos B$

LHS $= (a^2 - b^2 - c^2)\tan A + (c^2 + a^2 - b^2)$

$= (-2bc\cos A)\tan A + 2ca\cos B\tan B$

$= -2bc\sin A + 2ca\sin B$

$= 2c(a\sin B - b\sin A) = 0 =$ RHS $\left(\frac{a}{\sin A} = \frac{b}{\sin B}\right)$ proved.

Ex 5.(i) [QP 2011, 2013] Find the greatest angle of the triangle whose sides measure 3cm , 4 cm and 5 cm.

(ii) **[QP 2011]** The side of a triangle are 8cm, 10cm and 12cm. Find the greatest angle.

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

Worked out examples :

Ex 1. Prove that (i) $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

(ii) $a=$ **[QP 2009]** $b\cos C+c\cos B$

(iii) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

(iv) $\cos\dfrac{C}{2}=\sqrt{\dfrac{s(s-c)}{ab}}$

(v) $\sin A=2\dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}=\dfrac{2\Delta}{bc}$

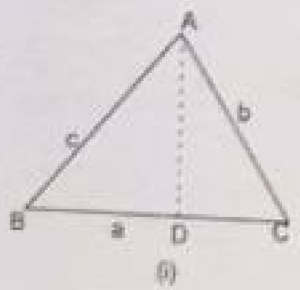
(vi) **[QP 2013]** $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

Proof: (i) Let us see the following figure. From geometry we have

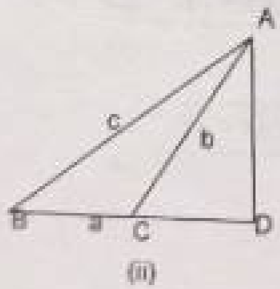
if $\angle C$ is acute, then $AB^2=BC^2+CA^2-2BC.CD$ ....(1)

if $\angle C$ is obtuse, then $AB^2=BC^2+CA^2+2BC.CD$ ....(2)

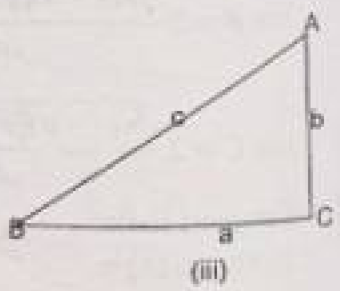
if $\angle C$ is right angled, then $AB^2=BC^2+CA^2$ ....(3)



(i)



(ii)



(iii)

From diagram it is clear that in (1) $CD=b\cos C$. Hence (1) can be written as

$c^2=a^2+b^2-2ab\cos C$

In (2) $\angle C$ is obtuse. So $CD=b\cos(\pi-C)=-b\cos C$. And (2) becomes

$c^2=a^2+b^2-2ab\cos C$

In (3) $c^2=a^2+b^2=a^2+b^2-2ab\cos C$ since $\cos C=0$

Hence for all values of C we have

$c^2=a^2+b^2-2ab\cos C$

or, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$ Proved.

(ii) In the above figure we see when $\angle C$ is acute,

$a=b\cos C+c\cos B$

In fig (ii) when $\angle C$ is obtuse, $a=c\cos B-b\cos(\pi-C)$

or, $a=c\cos B+b\cos C=b\cos C+c\cos B$

In fig (iii) when $\angle C$ is right angled, $a=c\cos B=c\cos B+0$

$=c\cos B+b\cos C$ (since $\cos C=0$)

$=b\cos C+c\cos B$

Hence for all values of C we have

$a=b\cos C+c\cos B$ Proved.

(iii) $2\sin^2\dfrac{A}{2}=1-\cos A=1-\dfrac{b^2+c^2-a^2}{2bc}$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a-b+c)(a+b-c)}{2bc}=\frac{(a+b+c-2b)(a+b+c-2c)}{2bc}$$

$$=\frac{(2s-2b)(2s-2c)}{2bc}\qquad 2s=a+b+c$$

$\Rightarrow \sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$ Proved.

(iv) $2\cos^2\dfrac{C}{2}=1+\cos C=1+\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{2ab+a^2+b^2-c^2}{2ab}$

Again $\angle BDC = \angle BAC = A$ Where A is acute

$\angle BCD = 180^\circ - \angle BAC = \angle 180^\circ - A$ as both angles based on the same segment BC. When A is obtuse Fig 2.10.1 (ii); $\sin\angle BDC = \sin A$ as quadrilateral ABCD is cyclic.

So in both cases, $\sin\angle BDC = \sin A$ ....(2)

From (1) and (2) $\sin BDC = \sin A = \dfrac{a}{2R}$

$$\Rightarrow \sin A = \frac{a}{2R} \Rightarrow 2R = \frac{a}{\sin A}$$

In fig(iii) $A = 90^\circ$ and $BC = a = 2R$. Now centre O lies on BC.

$$\text{Sin } A = \text{Sin } 90^\circ = 1 = \frac{a}{a} = \frac{a}{2R}$$

$$\Rightarrow \text{Sin } A = \frac{a}{2R} \Rightarrow 2R = \frac{a}{\sin A}; \text{ Hence in all cases, } \frac{a}{\sin A} = 2R$$

Similarly we get $\dfrac{b}{\sin B} = 2R$ and $\dfrac{c}{\sin C} = 2R$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Now we state some other formulae

2.7.2 Cosine formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \text{or, } a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \text{or, } b^2 = c^2 + a^2 - 2ca\cos B$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{or, } c^2 = a^2 + b^2 - 2ab\cos C$$

2.7.3 In any triangle

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$
$$c = a\cos B + b\cos A$$

2.7.4 Trigonometrical Ratios of half angles of a triangle in terms of the sides.

1. $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ **2.** $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}}$

3. $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$ **4.** $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

5. $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}}$ **6.** $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

7. $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$ **8.** $\tan\dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

9. $\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

Where s = semi-perimeter of the trinangle $= \dfrac{1}{2}(a+b+c)$

2.7.5 Sine of angle in terms of sides:

$$\sin A = 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc} = \frac{2\Delta}{bc}$$

$$\sin B = 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{ca} = \frac{2\Delta}{ca}$$

$$\sin C = 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{ab} = \frac{2\Delta}{ab}$$

2.7.6 Tangent Rule:

In any triangle

Exercise 2.6

1. Evaluate

(i) $\sin\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\tan\left(\frac{3\pi}{4}\right)$

(iii) $\sin\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ (iv) $\sec\sec^{-1}(-150^\circ)$ (v) $\sin\cos^{-1}(120^\circ)$

Ans: (i) $\frac{1}{2}$ (ii) $-\frac{\pi}{2}$ (iii) $\frac{-\sqrt{3}}{2}$ (iv) 150° (v) $-\frac{\pi}{6}$

2. Prove that

(i) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$ (ii) $\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{3}{4}$

(iii) $\cos^{-1}\frac{40}{41} = \tan^{-1}\frac{9}{40}$ (iv) $\sin^{-1}\frac{9}{41} = \cot^{-1}\frac{40}{9}$

3. Prove that

(i) $\tan^{-1}5 - \tan^{-1}3 = \tan^{-1}\frac{1}{8}$

(ii) $2\tan^{-1}\sqrt{x} = \cos^{-1}\frac{1-x}{1+x}$

(iii) $[\cos(\sin^{-1}x)]^2 = [\sin(\cos^{-1}x)]^2$

(iv) $\sin \operatorname{cosec}^{-1}\cot\tan^{-1}x = x$

(v) $\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\frac{xy \mp 1}{y \pm x}$

* * *

2.7 Properties of Triangles

Every triangle has three sides and three angles. In the triangle ABC the three angles are BAC, ABC, ACB are denoted by A, B, C. Similarly, the sides BC, CA, AB are denoted by small letters a, b, c respectively. circum circle is a circle passing through the vertices of a triangle. In this case centre of circle is called circum-centre and the radius is called circum-radius.

2.7.1 Sine Rule:

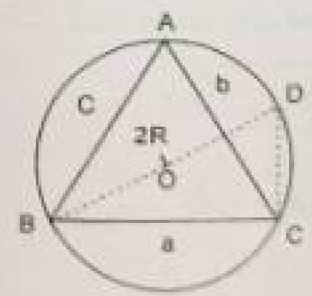
In a triangle ABC the sides are proportional to the sine of the corresponding angles

ie; $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

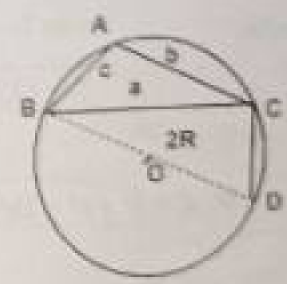
Let O be the centre and R be the radius of the cicum circle of ABC. If A is acute [fig(i)] or if A is obtuse [fig(ii)] join BO and produce it to meet at the point D on circumference. Now join DC. Since OB is radius therefore OB = R and BD = $2R$, $\angle$BCD=90°

From ΔBCD, we have,

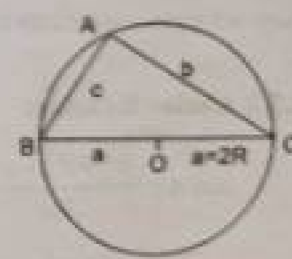
$\sin\angle BDC = \frac{BC}{BD} = \frac{a}{2R}$ ....(1)



(i)



(ii)



(iii)

3.1.4 Curvilineal Figures:

A curvilineal figure is a closed figure, either completely bounded by a closed curve as shown in figure (i) or bounded by a curve on one side only as

figure (i)

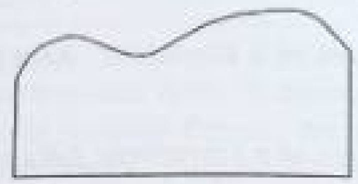


figure (ii)

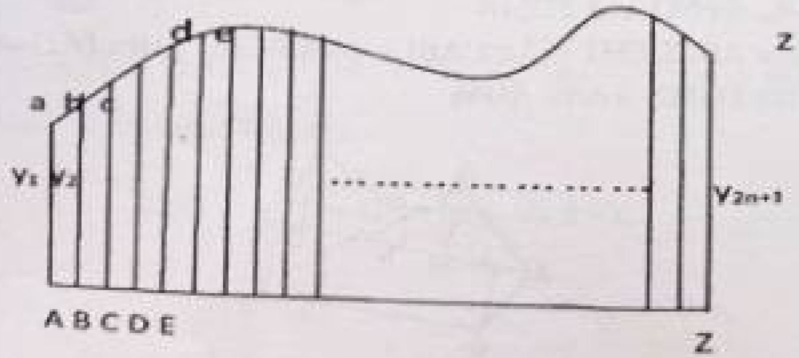
shown in figure (ii).

Area of a curvilineal figure can be obtained using a rule known as SIMPSON'S ONE THIRD RULE.

Now we proceed to observe how Simpson's rule works for different curvilineal figures.

Case I: Let us consider a curviilineal figure bounded by a curve on one side and three straight lines Aa, AZ and Zz on the other three sides where Aa and Zz are at right angles to AZ.The base line AZ is divided into *an even number of equal parts*, the thickness of each part being'd'. Next, at each of the points of divisions we draw lines perpendicular to AB called *ordinates*. 'd' is called the *common distance or c.d.*

Here,Aa,Bb,Cc,Dd,Ee .... are ordinates and d = AB = BC = CD = DE =.....



Since, AZ is divided into an even number of equal parts, the number of ordinates will be odd. Let, Aa = y_1, Bb = y_2, Cc = y_3, Dd = y_4, Ee = y_5, ... Zz = y_{2n+1}.

Now, the area of this curvilineal figure can be obtained by SIMPSON'S ONE THIRD RULE as follows: According to this rule,

Required Area = $\frac{\text{c.d}}{3}$ [(first ordinate + last ordinate) +2(sum of the remaining odd ordinates) + 4(sum of even ordinates)]

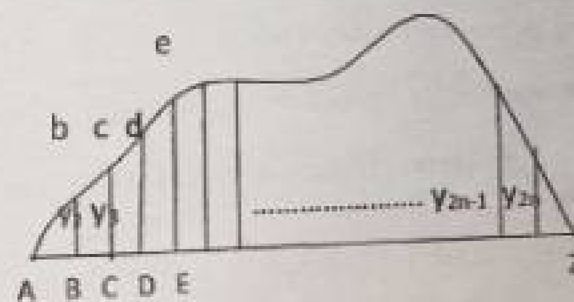
$$= \frac{d}{3}\ [\ (y_1 + y_{2n+1}) + 2(\ y_3 + y_5 + y_7 + \ldots + y_{2n-1}) + 4(\ y_2 + y_4 + y_6 + \ldots + y_{2n})]$$

Case II:A curvilneal figure formed by a curve and a straight line :

The base line AZ is divided into '$2n$' number of equal parts. $y_1, y_2, y_3, y_4, \ldots\ldots, y_{2n}, y_{2n+1}$ are ordinates, and 'd' is the common distance. Here the first ordinate y_1 and the last ordinate y_{2n+1} are zero. Therefore by Simpson's one third rule ,

The required area–

$$= \frac{d}{3}\ [2(y_3 + y_5 + y_7 + \ldots + y_{2n-1}) + 4(y_2 + y_4 + y_6 + \ldots + y_{2n})]$$



Case III: A curvilineal figure formed by a single closed curve : Here the ordinates are bb', cc', dd', ee'........ with the first and the last ordinate value zero, 'd' is the common distance.

3

MENSURATION

3.1 Area: Mensuration is the part of mathematics that deals in measurement of lengths, areas and volumes of regular as well as irregular figures. An elementary knowledge of geometry and algebra is sufficient for this purpose. There are three types of figures, one-dimensional, two-dimensional and three-dimensional. An one-dimensional figure has length, a two- dimensional figure has length as well as breadth and a three-dimensional figure has length, breadth and height. In this chapter we are going to discuss about the area of irregular figure and volume of some specific three-dimensional figures.

3.1.1. Some Important Formulae:

(a) Area of a Triangle with base and height:

Area of Δ ABC= $\frac{1}{2}bh$, b=base, h=height

(b) Area of a Triangle with side lengths:

Area of Δ ABC= $\sqrt{s(s-a)(s-b)(s-c)}$, $s=\frac{(a+b+c)}{2}$

(c) Area of a Right Angled Triangle:

Area of Δ ABC = $\frac{1}{2}bp$, b = base, p = perpendicular

(d) Area of an Equilateral Triangle:

$$\text{Area of } \Delta\,ABC = \frac{1}{2}ah = \frac{\sqrt{3}}{4}a^2, \quad h=\frac{\sqrt{3}}{2}a \text{ , } a = \text{side}$$

(e) Area of an Isosceles Triangle:

$$\text{Area of } \Delta\ ABC = \sqrt{s(s-a)(s-b)(s-c)}, b=c, s=\frac{a+b+c}{2}$$

$$=\frac{a}{4}\sqrt{4b^2-a^2}$$

(f) Area of a Parallelogram: Area of $\square$ABCD= $b.h$, b = base, h = height

(g) Area of a Rectangle: Area of $\square$ABCD = lb, l = length, b = breadth

(h) Area of a square: Area of $\square$ABCD = a^2

(i) Area of a Rhombus: Area of $\Diamond$ABCD = $\frac{1}{2}d_1d_2$, d_1, d_2 diagonals.

(j) Area of a Trapezium: Area of $\triangle$ ABCD = $\frac{1}{2}(a+b)h$, a,b parallel lines, h height.

(k) Area of a regular hexagon of side 'a'= $\frac{3\sqrt{3}}{2}a^2$

(g) Area of a regular octagon of side 'a'=$2(1+\sqrt{2})\,a^2$

3.1.2 RECTILINEAL FIGURES AND CURVILINEAL FIGURES:

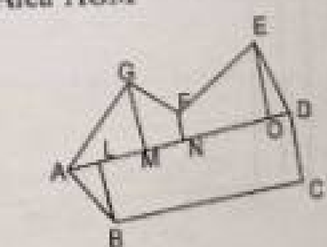
We have discussed about regular rectilineal figures in the last section. In this section we will discuss about irregular rectilinear figures and curvilineal figures.

3.1.3 Irregular Rectilineal Figures:

An irregular rectilineal figure is first divided into different parts, the area of each of these parts are obtained, the sum of all these areas will be the required area.

For Ex., in the figure ABCDEFG, first, a *base line* AD is drawn, then *offsets*, perpendiculars from the base line to the vertices B,E,F,G are drawn. As a result we will get three triangles, ABL, AGM, DEO and three trapezoids, BCDL, EFNO and FGMN.

$\therefore$ Area ABCDEFG = Area ABL+ Area BCDL +Area DEO+Area EFNO + Area FGMN +Area AGM



$$= \frac{b-c}{a}\frac{s(s-a)}{bc}+\frac{c-a}{b}\frac{s(s-b)}{ac}+\frac{a-b}{c}\frac{s(s-c)}{ab}$$

$$= \frac{s}{abc}\{(b-c)(s-a)+(c-a)(s-b)+(a-b)(s-c)\} = \frac{s}{abc}\times 0 = 0 =$$

RHS proved.

Ex 9. In a triangle ABC, $A = 60^\circ$, $B = 45^\circ$ and $c = 25$ c.m. Find the value of a.

Soln: $C = 180^\circ - (A+B)$

$= 180^\circ - 105^\circ$ (Given, $A = 60^\circ$, $B = 45^\circ$)

$= 75^\circ$

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow a = \frac{c}{\sin C}\sin A$

$\Rightarrow a = 25\frac{\sin 60^\circ}{\sin 75^\circ} \Rightarrow a = 25\frac{\sqrt{3}}{2}\times\frac{2\sqrt{2}}{\sqrt{3}+1} = \frac{25\sqrt{6}}{\sqrt{3}+1}$ cm Ans.

Ex 10. Given $a = 13$, $b = 14$, $c = 15$ then find the value of $\sin\frac{B}{2}$ and $\cos\frac{B}{2}$

Soln: We have, $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(13+14+15) = \frac{42}{2} = 21$

We have, $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{(21-15)(21-13)}{15\times 13}} = \frac{4\sqrt{3}}{\sqrt{195}}$

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{21(21-14)}{13\times 15}} = \sqrt{\frac{21\times 7}{195}} = \frac{7\sqrt{3}}{195}$ Ans.

Exercise 2.7

In a triangle prove that:

1. $c\cos\frac{1}{2}(A-B) = (a+b)\sin\frac{1}{2}C$

2. $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$

3. $a^2(\sin^2 B - \sin^2 C) + b^2(\sin^2 C - \sin^2 A) + c^2(\sin^2 A - \sin^2 B) = 0$

4. $(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0$

5. $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$

6. If $C = 60^\circ$ then prove that $\frac{1}{a+c}+\frac{1}{b+c} = \frac{3}{a+b+c}$

7. In a triangle, if $a = 5$, $b = 13$ and $C = 90^\circ$ then find the area of the triangle.

8. In a triangle, $C = 60^\circ$ then prove that $2a - b = 2c\cos B$

9. In a triangle, $B = 90^\circ$ then prove that $\tan\frac{A}{2} = \sqrt{\frac{b-c}{b+c}}$

10. In a triangle, prove that $2(abc\cos A + bc\cos A + ca\cos B) = a^2+b^2+c^2$

* * *

(iii) **[QP 2014]** In any triangle the sides are a = 7, b = 5, c = 8. Find A.

Soln: (i) Let $a = 3$cm, $b = 4$ cm, $c = 5$cm

Here c is the largest side

$\therefore$ angle C will be largest and $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$\Rightarrow \cos C = \frac{3^2+4^2-5^2}{2.3.4} = 0$$

$$\Rightarrow \cos C = 0 = \cos 90^\circ$$

$$\therefore C = 90^\circ \quad \text{Ans.}$$

(ii) Let $a = 8$cm, $b = 10$cm, $c = 12$cm

Here c is the largest side.

$\therefore$ angle C will be largest and $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$\Rightarrow \cos C = \frac{8^2+10^2-12^2}{2.8.10} = \frac{1}{8}$$

$$\therefore C = \cos^{-1}\frac{1}{8} \quad \text{Ans}$$

(iii) Let $a = 7$cm, $b = 5$ cm, $c = 8$cm

Since $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ hence $\cos A = \dfrac{5^2+8^2-7^2}{2.5.8} = \dfrac{1}{2} = \cos 60^\circ$

$$\Rightarrow A = 60^\circ$$

Ex 6. (i) $a(b\cos C - c\cos B) = b^2 - c^2$

(ii) $a\sin A - b\sin B = c\sin(A-B)$

(iii) **[QP 2015]** $b\cos B + c\cos C = a\cos(B-C)$

Soln: (i) We have, $a = b\cos C + c\cos B$

Now LHS $= a(b\cos C - c\cos B)$

$= (b\cos C + c\cos B)(b\cos C - c\cos B)$

$= b^2\cos^2 C - c^2\cos^2 B$

$= b^2(1-\sin^2 C) - c^2(1-\sin^2 B)$

$= (b^2-c^2) + (c^2\sin^2 B - b^2\sin^2 C)$

$= b^2 - c^2 + 0$ since $c\sin B = b\sin C$ (from $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$)

$=$ RHS proved.

(ii) LHS $= a\sin A - b\sin B = 2R\sin A\sin A - 2R\sin B\sin B$

$= 2R(\sin^2 A - \sin^2 B)$

$= 2R\sin(A+B)\sin(A-B)$

$= 2R\sin(\pi - C)\sin(A-B)$

$= 2R\sin C\sin(A-B)$

$= c\sin(A-B) =$ RHS proved.

(iii) LHS $= b\cos B + c\cos C$

$= 2R\sin B\cos B + 2R\sin C\cos C$

$= R(\sin 2B + \sin 2C)$

$= 2R\sin(B+C)\cos(B-C)$

$= 2R\sin(\pi - A)\cos(B-C)$

$= 2R\sin A\cos(B-C)$

$= a\cos(B-C) =$ RHS proved.

Ex 7. If $a\cos^2\dfrac{C}{2} + c\cos^2\dfrac{A}{2} = \dfrac{3b}{2}$ then prove that a, b, c are in A.P.

Proof : Given $a\cos^2\dfrac{C}{2} + c\cos^2\dfrac{A}{2} = \dfrac{3b}{2}$

$$\Rightarrow 2a\cos^2\frac{C}{2} + 2c\cos^2\frac{A}{2} = 3b$$

$$\Rightarrow a(1+\cos C) + c(1+\cos A) = 3b$$

$$\Rightarrow a + c + (a\cos C + c\cos A) = 3b$$

$$\Rightarrow a + c + b = 3b \quad \Rightarrow a + c = 2b$$

$$\Rightarrow b - a = c - b$$

$\therefore a, b, c$ are in A.P. proved.

Ex 8. In a triangle ABC, prove that

$$\frac{b-c}{a}\cos^2\frac{A}{2} + \frac{c-a}{b}\cos^2\frac{B}{2} + \frac{a-b}{c}\cos^2\frac{C}{2} = 0$$

Proof: LHS $= \dfrac{b-c}{a}\cos^2\dfrac{A}{2} + \dfrac{c-a}{b}\cos^2\dfrac{B}{2} + \dfrac{a-b}{c}\cos^2\dfrac{C}{2}$

$$\text{Area} = \frac{14.6}{3}\{(18+14)+2(26+20+30+28)+4(22+24+26+34+24)\}$$

$$= \frac{14.6}{3}(32+208+520) = 3698.6 \text{ sq mt}$$

Exercise 3.1

1. The ordinates of a curve measures 5, 6, 8, 7, 4, 3, 2 ft respectively and their common distance is 1 foot. Find the area of the figure. (Ans. 31.67 sq. ft)
2. After flood havoc a riverside land is under erosion and appears like a curvilineal figure whose ordinates measure 9.5, 11, 13, 12.6, 10.4, 13, 15.5, 17, 14.9, 12.6 and 7.7 m with base length 7.3 m. Apply Simpson's rule to find the approximate area of the land. (Ans. 94.8 sq. m)
3. A river is 32 m wide. The depth 'd' in metres at a distance 'x' m from one bank is given by the following table:

x:	0	4	8	12	16	20	24	28	32
d:	0	10	20	25	30	41	44	26	10

Find the approximate cross-section of the river. (Ans. 808 sq. m)

4. The breadth of a river at some place is 40 m. If 'd' m be the depth of water at distances 'x' m, as given below, from one of its banks, find the cross-sectional area of the river

x:	0	5	10	15	20	25	30	35	40
d:	0	10	24	28	32	41	40	15	5

(Ans. 921.67 sq. m)

*****

VOLUME AND SURFACE AREAS OF REGULAR SOLIDS:

In the last two sections we had discussed about plane figures. In this section we will discuss about solids. Solids are three dimensional figures. Here, we will obtain the volume and surface areas of the following solids: Cuboids, Prism, Cylinder, Sphere, Cone and Pyramid.

3.2 Cuboid: A Cuboid is a solid bounded by six rectangular faces. It has three dimensions, length, breadth and height. The face on which the Cuboid rests is called its *base*, the four faces which meet the base are called *lateral faces* and the face opposite to the base is the *top*.

Consider a Cuboid with length 'l', breadth 'b' and height 'h'.

(a) Volume : V= Area of the base x height
$= lbh$ cub. unit

(b) Lateral Surface Area = Sum of the Area of the lateral faces
$= 2(l+b)h$ sq. unit

(c) Total Surface Area = Lateral Surface Area +Area of Base and Top
$= 2(lb + bh + lh)$ sq. unit

(d) Length of a diagonal = $\sqrt{l^2+b^2+h^2}$ unit

(e) Total Length of the Edges = $4(l+b+h)$ unit

A *Cube* is a special case of a Cuboid. All the edges of a Cube are equal. For a Cube of edge length a unit

(a) Volume : V = a^3 cu unit

(b) Lateral Surface Area = $4a^2$ sq unit

(c) Total Surface Area = $6a^2$ sq unit

(d) Length of a diagonal = $\sqrt{3}\,a$ unit

(e) Total Length of the Edges = $12a$ unit

Worked out examples:

Ex 1. The dimensions of a cuboid are in the ratio 1:2:3 and its total surface area is 352 sq. m. Find the dimensions and also the volume of the Cube.

Solution: The dimensions are in the ratio 1:2:3. Let the dimensions be x, $2x$ and $3x$.

According to question, Total Surface Area =352 sq m

$\Rightarrow 2(x.2x + 2x.3x + x.3x) = 352$ sq m

$\Rightarrow 22x^2 = 352$ sq m

$\Rightarrow x^2 = 16$ sq. m

$\Rightarrow x = 4$ m

$\therefore$ The dimensions are: $x = 4$m, $2x = 8$m, $3x = 12$m

And, required Volume $= x.2x.3x = 6x^3 = 6.4^3$ cu.m = 384 cu m

Ex 9. A river is 50 metre wide. The depth of water d metres at a distance x metres from one bank is given by the following table-

x:	0	5	10	15	20	25	30	35	40	45	50
d:	0	2	4	5	6	8	10	9	3	3	2

Find the area of cross section of the river water.

Solution: Here $d = 5$.

$$\text{Area} = \{(0 + 2) + 2(4 + 6 + 10 + 3) + 4(2 + 5 + 8 + 9 + 3)\}$$
$$= (2 + 46 + 108) = 260 \text{ sq metre.}$$

Ex 10. The following offsets were taken from a survey line to a hedge:

Distance from one end of a survey line (in mt) :	0	6	12	18	24	30	36	42	48	54	60
Offset(in mt):	3	3.4	4.3	3.9	2.7	2.3	2	1.8	1.5	2.1	2

Find the area between the survey lie and the hedge.

Solution: Here $d = 6$.

$$\text{Area} = \frac{6}{3}\{(3 + 2) + 2(4.3 + 2.7 + 2 + 1.5) + 4(3.4 + 3.9 + 2.3 + 1.8 + 2.1)\}$$
$$= 2(5 + 21 + 54) = 160 \text{ sq metre.}$$

Ex 11. The velocity of a train which starts from rest is given by the following table the time being reckoned in minutes from the rest and the speed in km/hr.

Time:	0	2	4	6	8	10	12	14	16	18	20
Speed:	0	10	18	25	29	32	20	11	5	2	0

Estimate by using Simpson's rule approximately the distance run in 20 min.

Soln: Distance = Speed (km/min) × Time (min)

We rewrite the table for speed in minute.

Time:	0	2	4	6	8	10	12	14	16	18	20
Speed:	0	$\frac{10}{60}$	$\frac{18}{60}$	$\frac{25}{60}$	$\frac{29}{60}$	$\frac{32}{60}$	$\frac{20}{60}$	$\frac{11}{60}$	$\frac{5}{60}$	$\frac{2}{60}$	0

$d = 2$

$$\text{Distance} = \frac{2}{3}\{(0 + 0) + 2(\frac{18}{60} + \frac{29}{60} + \frac{20}{60} + \frac{5}{60}) + 4(\frac{10}{60} + \frac{25}{60} + \frac{32}{60} + \frac{11}{60} + \frac{2}{60})\}$$
$$= \frac{2}{3}(2\times\frac{72}{60} + 4\times\frac{80}{60}) = \frac{2}{3}\times\frac{464}{60} = 5.15\text{km}$$

Ex 12. [2016] A river is 80 ft wide. The depth of water d at a distance x ft from one bank is given by the following table-

x:	0	10	20	30	40	50	60	70	80
d:	0	40	75	94	121	153	142	86	31

Find the area of cross section of the river water.

Solution: Here $d = 10$.

$$\text{Area} = \frac{10}{3}\{(0 + 31) + 2(75 + 121 + 142) + 4(40 + 94 + 153 + 86)\}$$
$$= (31 + 676 + 1492) = 7330 \text{ sq ft.}$$

Ex 13. [2016] The length of a line is 1344 cm and at equal distance along it, the following ordinates were taken to an irregularly curved fence:
76, 83, 87, 93, 112, 123, 137, 146, 139, 127, 116, 97, 82, 79, 64
Find the area included between the extreme offsets, the fence and the base line.

Soln: There are 15 offsets. $d = 1344/14 = 96$ cm

$$\text{Area} = \frac{96}{3}[(76 + 64) + 2(87 + 112 + 137 + 139 + 116 + 82) + 4(83 + 93 + 123 + 146 + 127 + 97 + 79)]$$
$$= 32\,(140 + 1346 + 2992) = 143296 \text{ sq cm.}$$

Ex 14. [2014, 2017] Find area of the curvilinear figure whose ordinates measure 18, 22, 26, 24, 20, 26, 30, 34, 28, 24, 14 mt and whose ordinates are at a distance of 14.6 mt from each other.

Soln: There are 11 offsets. $d = 14.6$ mt

Solution: Given in meters,

$y_1=2.0$, $y_2=1.8$, $y_3=3$, $y_4=3.4$, $y_5=3.9$, $y_6=1.5$, $y_7=1.2$; $d=6$

According to Simpson's one third rule,

$$\text{Required area} = \frac{d}{3}[(y_1+y_7)+2(y_3+y_5)+4(y_2+y_4+y_6)]$$

$$= \frac{6}{3}[(2.0+1.2)+2(3+3.9)+4(1.8+3.4+1.5)] \text{ sq.m}$$

$$= 2[3.2+2\times 6.9+4\times 6.7] \text{ sq.m} = 87.6 \text{ sq.m}$$

Ex 5. [2011, 2014, 2015] : An irregular plot has the following offsets(l) measured from one end at equal distances:

x :	0	12	24	36	48	60	72	84	96	108	120
l :	53	52	47	49	53	63	58	61	52	49	48

Find the area of the plot.

Solution : Given, $y_1=53$, $y_2=52$, $y_3=47$, $y_4=49$,
$y_5=53$, $y_6=63$, $y_7=58$, $y_8=61$,
$y_9=52$, $y_{10}=49$, $y_{11}=48$ and $d=12$

According to Simpson's one third rule,

$$\text{Required area} = \frac{d}{3}[(y_1+y_{11})+2(y_3+y_5+y_7+y_9)+4(y_2+y_4+y_6+y_8+y_{10})]$$

$$= \frac{12}{3}[(53+48)+2\times(47+53+58+52)+4\times(52+49+63+61+49)]$$

$$= 4[101+2\times 210+4\times 274] = 6468 \text{ sq unit}$$

Ex 6. Find by Simpson's rule the area of a curvilineal figure whose ordinates are 18, 22, 26, 24, 20, 26, 30, 34, 28, 24, 14 ms and whose base is 146 m.

Solution: Given ordinates (m) are :

$y_1=18$, $y_2=22$, $y_3=26$, $y_4=24$, $y_5=20$, $y_6=26$,
$y_7=30$, $y_8=34$, $y_9=28$, $y_{10}=24$, $y_{11}=14$.

The base length = 146ms, number of ordinates =11,

Number of divisions =10 and $d=\frac{146}{10}=14.6$ m

By Simpson's rule, Area

$$= \frac{d}{3}[(y_1+y_{11})+2(y_3+y_5+y_7+y_9)+4(y_2+y_4+y_6+y_8+y_{10})]$$

$$= \frac{14.6}{3}[(18+14)+2(26+20+30+28)+4(22+24+26+34+24)] \text{ sq. m}$$

$$= \frac{14.6}{3}[32+2\times 104+4\times 130] \text{ sq. m} = 3698.67 \text{ sq. m}$$

Ex 7. [2013] : The cross sectional area of a tunnel is as follows:

Distance from one end :

	0	3	6	9	12	15	18
Areas :	27.9	30.6	33.8	32.4	30.7	27.9	26.1

Find the volume of the solid.

Solution: Areas are considered as ordinates,

$y_1=27.9$, $y_2=30.6$, $y_3=33.8$, $y_4=32.4$, $y_5=30.7$ $y_6=27.9$, $y_7=26.1$; $d=3$

According to Simpson's one third rule,

$$\text{Required area} = \frac{d}{3}[(y_1+y_7)+2(y_3+y_5)+4(y_2+y_4+y_6)]$$

$$= [(27.9+26.1)+2(33.8+30.7)+4(30.6+32.4+27.9)] \text{ cub.unit}$$

$$= [54+129+363.6] \text{ cub unit} = 546.6 \text{ cub. unit}$$

Ex 8. The cross section of a tree is A square inches at a distance x inchesfrom one end. Corresponding values of A and x are-

x:	10	30	50	70	90	110	130	150	170
A:	120	123	122	129	131	135	142	156	177

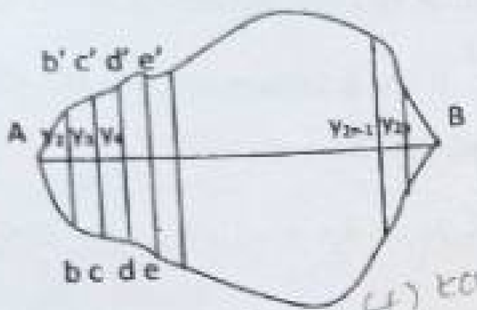
Find the volume of the tree in cubic inches between $x=10$ and $x=170$.

Solution: Here $d=20$.

$$\text{Volume} = \frac{20}{3}\{(120+177)+2(122+131+142)+4(123+129+135+156)\}$$

$$=(297+790+2172)=21726.6 \text{ cu in.}$$

Required area $= \frac{d}{3}[2(y_1+ y_3+ y_5+.....+ y_{2n-1}) + 4(y_2+ y_4+ y_6+.....+ y_{2n})]$



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Worked out examples:

Ex 1. [2013] : A river is 80ft wide. Its depth at a distance of 'x' ft from one bank is 'd' ft and is given by the following table below :

x	0	10	20	30	40	50	60	70	80
d	0	40	75	94	121	153	142	86	31

Find the approximate cross sectional area of the river.

Solution: Given(ft), $y_1=0$, $y_2=40$, $y_3=75$, $y_4=94$, $y_5=121$, $y_6=153$, $y_7=142$, $y_8=86$, $y_9=31$ and $d=10$

According to Simpson's one third rule,

Required area $= \frac{d}{3}[(y_1+y_9)+2(y_3+y_5+y_7)+4(y_2+y_4+y_6+y_8)]$

$= \frac{10}{3}[(0+31)+2(75+121+142)+4(40+94+153+86)]$ sq.ft

$= \frac{10}{3}[31 + 2.338 + 4.373]$ sq.ft $= 7330$ sq.ft

Ex 2. [2009] : Find by Simpson's rule the area of the curvilinear figure whose ordinates measure 18, 22, 26, 24, 20, 26, 30, 34, 28, 24, 14 metres and whose base is 150 metres.

Solution : Given ordinates are(in ms):

$y_1 = 18$, $y_2 = 22$, $y_3 = 26$, $y_4 = 24$, $y_5 = 20$, $y_6 = 26$,

$y_7 = 30$, $y_8 = 34$, $y_9 = 28$, $y_{10} = 24$, $y_{11} = 14$.

The base length = 150 ms. number of ordinates = 11.

$\therefore$ number of divisions = 10 and $d = \frac{150}{10}$ m = 15 m.

By Simpson's rule, required area

$= \frac{d}{3}[(y_1 + y_{11}) + 2(y_3 + y_5 + y_7 + y_9) + 4(y_2 + y_4 + y_6 + y_8 + y_{10})]$

$= \frac{15}{3}[(18+14) + 2(26+20+30+28) + 4(22+24+26+34+24)]$ sq.m

$= 5[32 + 2.104 + 4.130]$ sq. m $= 3800$ sq. m

Ex 3. [2010] : A river is 100 mts. wide, the depth 'd' in metres at a distance 'x' metres from one bank is given by the following table:

x :	0	10	20	30	40	50	60	70	80	90	100
d :	5	30	42	51	60	76	58	47	32	15	6

Find approximately the cross-sectional area of the river.

Solution: The given ordinates (in m) are: $y_1=5$, $y_2=30$, $y_3=42$, $y_4=51$, $y_5=60$, $y_6=76$, $y_7=58$, $y_8=47$, $y_9=32$, $y_{10}=15$, $y_{11}=6$ and $d=10$m

$\therefore$ By Simpson's one third rule,

Required Area $= \frac{d}{3}[(y_1+y_{11}) + 2(y_3+y_5+y_7+y_9) + 4(y_2+y_4+y_6+y_8+y_{10})]$

$= \frac{10}{3}[(5+6) + 2(42+60+58+32) + 4(30+51+76+47+15)]$ sq. m

$= \frac{10}{3}[11 + 2 \times 192 + 4 \times 219]$ sq. m

$= 4236.67$ sq. m

Ex 4. [2011] : The following offsets were taken from a survey line to a hedge:

Distance from survey line (m) :	0	6	12	18	24	30	36
offsets (m) :	2.0	1.8	3	3.4	3.9	1.5	1.2

Find the area between the survey line and the hedge.

Exercise 3.3

1. Find the edge of a cube whose volume is same as that of a right prism 4 dm high, standing on a triangular base whose sides are 5 cm, 5 cm and 8 cm. (Ans. 7.83 cm)
2. A right prism stands on a triangular base whose sides are 18 cm, 20 cm and 34 cm. If the height is 10 cm, find the volume and whole surface area of the prism. (Ans: 1440 cub.cm,1008 sq.cm)
3. The lateral surface area of an equilateral triangular prism of height 24 inch is 504 sq.inch. Find the volume of the prism. (Ans: $294\sqrt{3}$ cub.cm)
4. The base of a right prism is an equilateral triangle of side 23 cm and its height is 38 cm. Find the volume. (Ans. 8704.42 cub.cm)-

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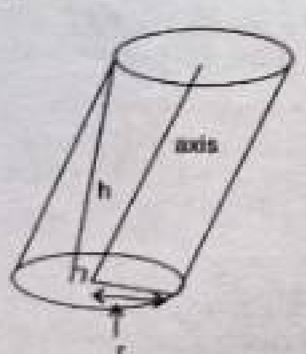
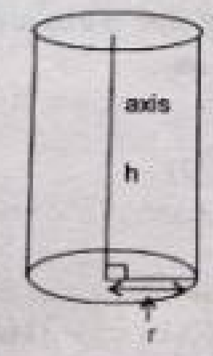
3.4 Cylinder: Cylinder is a solid bounded by a curved surface and two circular planes. Circular pillars, pipes, garden rollers, gas cylinders, iron rods are some Examples of cylinders. Cylinder is considered to be a special case of a prism. Because, if we go on increasing the number of sides at the ends of a prism, at one time they will resemble circles and the figure will be a cylinder. Cylinders like pipe, rubber tube are hollow inside. They are called *Hollow Cylinders.*

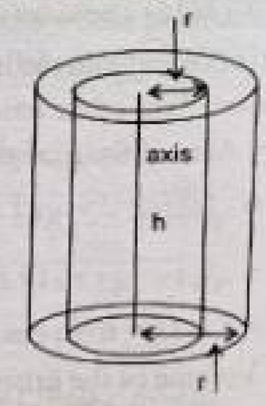
The line joining the mid points of the circular planes is called *axis* of the cylinder. When the axis is perpendicular to the circular planes, the cylinder is a *Right Circular Cylinder*, otherwise, it is not a right circular cylinder.



Circular Cylinder



Right Circular Cylinder



Hollow Cylinder

Volume and Surface Area:

(a) For any Circular Cylinder: Volume = Area of the base × height
$= Ah = \pi r^2 h$ cu unit

(b) For a Right Circular Cylinder:

(i) Volume = Area of the base × height = Ah = $\pi r^2 h$ cu unit

(ii) Lateral Surface Area = Area of the curved surface = $2\pi rh$ sq unit

(iii) Total Surface Area = Lateral Surface Area + Area of the two ends
$= 2\pi rh + 2\pi r^2$ sq.unit $= 2\pi r(r + h)$ sq unit

(c) For a Hollow Cylinder:

(i) Area of one end = A = $\pi (R^2 - r^2)$ sq.unit

(ii) Volume = Area of the base × height = $\pi (R^2 - r^2)\, h$ cub.unit

(iii) Lateral Surface Area = External surface area + Internal surface area
$= 2\pi(R + r)h$ sq.unit

(iv)Total Surface Area = Lateral surface area + Area of the two ends
$= 2\pi(R + r)h + 2\pi (R^2 - r^2)$ sq.unit

Worked out examples:

Ex 1. Find the volume and the curved surface of a right circular cylinder whose height is 12 metres and the diameter of the base 6 metres.

Solution: Given, h = 12m, diameter = 6m, therefore, radius r = 3m

∴ Required Volume of the cylinder = $\pi r^2 h$
= 3.1416 × 3 × 3 × 12 cub.m
= 339.29 cub.m

∴ Required Curved Surface of the cylinder = $2\pi rh$ = 2x3.1416x3x12 sq.m
= 226.20 sq.m

Ex 2. The height of a cylinder is 10 cm and the ratio of its volume to its lateral surface is 3:2. Find the radius of the cylinder.

Solution: According to question, h = 10 cm and
Volume : Lateral Surface Area = 3:2

$$\Rightarrow \frac{\pi r^2 h}{2\pi rh} = \frac{3}{2} \Rightarrow r = 3 \text{ cm}$$

Ex 3. Find the radius of a cylinder 14 cm high whose volume is equal to that of a cube having an edge of 11 cm.

Ex 4. The base of a right triangular prism is an equilateral triangle. The height of the prism is 13 unit and its volume is $832\sqrt{3}$ cub. unit. Find the surface area of the prism [$\sqrt{3}$ =1.732].

Solution: Given, volume of the prism = $832\sqrt{3}$ cub. unit, $h = 13$ unit

We have, $V = Ah$, A = area of an equilateral triangle

$$\Rightarrow V = \frac{\sqrt{3}}{4}a^2 h \Rightarrow 832\sqrt{3} = \frac{\sqrt{3}}{4}a^2 \times 13 \Rightarrow a^2 = 256 \Rightarrow a = 16 \text{ unit}$$

Total surface area of the prism = perimeter $\times$ height + 2A

$= 3ah + 2A$

$= (3\times 16\times 13 + 2\times \frac{\sqrt{3}}{4}\times 16\times 16)$ sq. unit

$= (624 + 221.70)$ sq. unit $= 845.70$ sq. unit

Ex 5. A concrete pillar has the base in the form of a regular hexagon of side 25 cm. Find the volume of the pillar if it is of height 4.6 metres.

Solution: Volume of the pillar: $V = Ah$, A= area of a regular hexagon, h = height

$$\Rightarrow V = \frac{3\sqrt{3}}{2}a^2 h$$

Given, $a = 25$ cm, $h = 4.6$ m = 460 cm

Therefore required volume $= \frac{3\sqrt{3}}{2}\times 25\times 25\times 460$ cub.cm = 746946.91 cub. cm

Ex 6. [2011] The base of a prism of height 24 cm is a trapezoid whose parallel sides are 26 cm and 34 cm. If the distance between the parallel sides be 18 cm, find the volume of the prism.

Solution: Volume of a prism = Ah, A = area of the base, h = height = 24 cm

Here, base is a trapezoid. Therefore,

$A = \frac{1}{2}(a+b).d$, $a = 26$cm, $b = 34$cm, $d = 18$cm.

$= \frac{1}{2}(26 + 34)\times 18$ sq. cm = 540 sq. cm

Therefore, required volume = 540×24 cub. cm = 12960 cub. cm

Ex 7. Find the cubical contents of a pillar 4 m high, if the mean section of the pillar is a regular hexagon of side 30 cm.

Solution: Volume of the pillar: $V = Ah$, A = area of the mean section of the pillar, h = height Given $a = 30$ cm, $h = 4$ m =400 cm

$$V = \frac{3\sqrt{3}}{2}a^2 h = \frac{3\sqrt{3}}{2}\times 30\times 30\times 400 \text{ cub.cm} = 935307.44 \text{ cub.cm}$$

Ex **8. [2014]** Find the whole surface area of a right prism whose height is 75 cm and whose base is a regular octagon of side 12 cm.

Solution: Area of the base = area of the octagon of side 12 cm

$= 2\times 12^2(1+\sqrt{2}) = 696.96$ sq cm

It has 2 bases. So area of two bases is 2×696.96 sq cm = 1393.9 sq cm

The lateral surface is a combination of 8 12×75 rectangles.

Hence lateral surface area = $8\times 12\times 75$ sq cm = 7200 sq cm

Total surface area = (7200 + 1393.9) sq cm = 8393.9 sq cm

Ex 9. [2015] The area of the lateral surface of a right prism is 80 sq mt. If the base of the prism be a square of side 4 mt, find the height of the prism and its volume.

Solution: Area of the lateral surface = perimeter of the base $\times$ height

$\Rightarrow 80 = 16\times$ height $\Rightarrow h = 5$ mt.

Volume = Area of the base $\times$ height = $16\times 5 = 80$ cu mt.

Ex 10. [2015] The sides of a triangular prism are 25 cm, 51 cm, and 52 cm and height is 60 cm. Find the length of the side of a cube of equivalent volume.

Solution: Area of the base of the prism

= Area of the triangle

$= \sqrt{64(64-25)(64-51)(64-52)}$ here $2s = 25 + 51 + 52 = 128$

$= \sqrt{64\times 3\times 13\times 13\times 4\times 3}$

$= 8\times 13\times 6$ sq cm = 624 sq cm

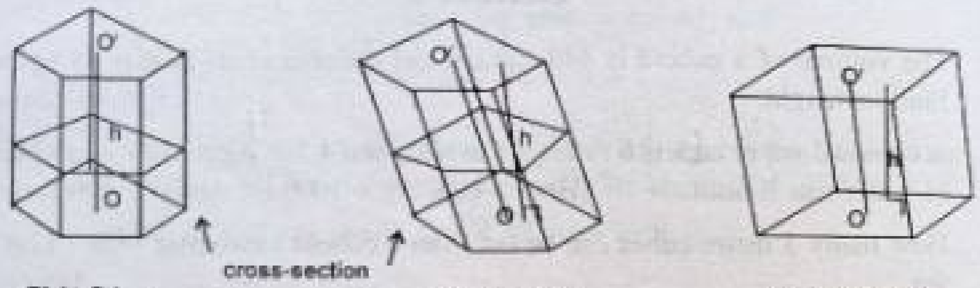
Volume of the prism = 624×60 cu cm = 37440 cu cm

= Volume of the cube of side $a = a^3$ $a = 33.45$ cm

3.3 Prism: A Prism is a solid whose sides are parallelograms and whose ends lie in parallel planes. The end on which the prism rests is called its *Base*. The perpendicular distance between the two ends is called its *height*. The straight line joining the mid points of the two ends is the *axis* of the prism.

RIGHT PRISM: A prism is called a right prism if its edges formed by adjacent side faces are perpendicular to one another.

OBLIQUE PRISM: A prism is called an oblique prism if its edges formed by adjacent side faces are not perpendicular to one another.



Right Prism Oblique Prism Parallelopiped

In all the there figure, OO' = axis, h = height

PARALLELOPIPED: When the ends of a prism are parallelograms, the prism is called a parallelepiped. *A rectangular parallelepiped is a cuboid because its ends are rectangles.*

A prism is named according to the shape of its base. For example. triangular prism, equilateral triangular prism, pentagonal prism, hexagonal prism etc. A prism is called *Regular* if its base is a regular figure, that is, all of its sides are equal.

Volume and Surface Area :

(a) Volume = Area of the base × height = Ah

(b) Volume = Area of the cross-section × height

(c) Lateral Surface Area = Perimeter of the base × height

(d) Total Surface Area = Lateral Surface Area + Area of the bases

Worked out examples:

Ex 1. The base of a right prism is an equilateral triangle of side 7 inch and the height of the prism is 2 ft. Find the volume of the prism.

Solution: Volume of the prism $=Ah$, Given, $a=7$inch, $h=2$ft $=2\times12$ inch $=24$ inch

Here, A = Area of the equilateral triangle $=\dfrac{\sqrt{3}}{4}a^2=\dfrac{\sqrt{3}}{4}\times7\times7$ sq. inch

$\therefore$ Required volume $=\dfrac{\sqrt{3}}{4}\times7\times7\times24$ cub. inch $=509.22$ cub. inch

Ex 2. [2009] Find the volume of a triangular prism whose height is 3 cm and the lengths of sides of triangular base are 12 cm, 13 cm and 5 cm.

Solution: Volume of the Prism $=Ah$, $A=$ Area of the triangle

Given, $h=3$ cm, $a=12$ cm, $b=13$ cm, $c=5$ cm

$$\therefore A=\sqrt{s(s-a)(s-b)(s-c)},\quad s=\frac{a+b+c}{2}=\frac{12+13+15}{2}=15\text{ cm}$$

$$=\sqrt{15(15-12)(15-13)(15-5)}\text{ sq. cm}=30\text{ sq. cm}$$

Therefore Required Volume $=30\times3$ cub. cm $=90$ cub. cm

Ex 3. [2009] The sides of a triangular prism are 25 cm, 51 cm, 52 cm and height is 60 cm. Find the volume of the prism and the length of the side of a cube of equal volume.

Solution: A triangular prism is given. Let the sides of the triangle be, $a=25$ cm, $b=51$ cm, $c=52$ cm and height: $h=60$ cm.

$$\text{Area of the base: } A=\sqrt{s(s-a)(s-b)(s-c)},\ s=\frac{(a+b+c)}{2}=\frac{(25+51+52)}{2}=64$$

$$=\sqrt{64(64-25)(64-51)(64-52)}\text{ sq. cm}$$

$$=\sqrt{64\times39\times13\times12}\text{ sq. cm}=624\text{ sq. cm}$$

$\therefore$ Volume of the prism: $V=Ah=624\times60$ cub. cm $=37440$ cub. cm

Let 'l' be the length of a side of the cube of equal volume.

$$\therefore l=\sqrt[3]{37440}\text{ cm}=33.45\text{ cm.}$$

Ex 2. A solid cube is cut into two cuboids of equal volumes. Find the ratio of the total surface area of the given cube and that of one of the cuboids.

Solution: Let the edge of the cube be 'a' unit.

$\therefore$ Total surface area of the cube: $S = 6a^2$ sq. unit

The cube is cut into two cuboids of equal volumes, therefore, for each cuboid

length $= a$, breadth $= a$, height $= \frac{a}{2}$

$\therefore$ Total surface area of one cuboid:

$$S_1 = 2(\ a\times a + a\times\frac{a}{2} + a\times\frac{a}{2})\text{sq. unit} = 4a^2 \text{ sq. unit}$$

$\therefore$ Required Ratio, $S : S_1 = 6a^2 : 4a^2 = 3:2$

Ex 3. Find the length of a rectangular solid whose volume measures 41cub.ft 432cub.inch, breadth 2ft. 9inch and depth 3ft 4inch.

Solution: Here, Volume: V= 41 cub ft 432 cub inch

$$= 41\frac{432}{12\times12\times12}\text{ cub.ft}$$

$$= \frac{165}{4}\text{ cub.ft.}$$

Breadth: $b = 2\text{ ft }9\text{ inch} = 2\frac{9}{12}\text{ ft} = \frac{11}{4}\text{ ft}$, $h = 3\text{ ft }4\text{ inch} = 3\frac{4}{12}\text{ ft} = \frac{10}{3}\text{ ft}$

$$\therefore \text{Length} = \frac{V}{b\times h} = \frac{165}{4}\times\frac{4}{11}\times\frac{3}{10}\text{ ft} = \frac{9}{2}\text{ ft} = 4\text{ft }6\text{ inch.}$$

Ex 4. [2011]. What is the length of the greatest rod that can be placed in a room whose length is 30 ft, breadth 24 ft and height 18 ft?

Solution: Dimensions of the room: $l = 30$ ft, $b = 24$ ft, $h = 18$ ft. The room resembles

a cuboid. $\therefore$ Length of the greatest rod = Length of the diagonal

$$= \sqrt{(l^2+b^2+h^2)} = \sqrt{(30^2+24^2+18^2)} = 42.43\text{ ft.}$$

Ex 6. [2012] Find the cost at the rate of Rs.100 per cubic meter of digging a pit of square size 2m × 2m and depth 4m.

Solution: Size of the pit, length=2m, breadth=2m, height=4m.

$\therefore$ Volume of the pit = $2\times2\times4$cub.m = 16cub.m

Rate of digging = Rs.100 per cub.m

$\therefore$ Total cost of digging = Rs.100 ×16 = Rs.1600

Exercise 3.2

1. The volume of a cuboid is 440 cub.cm and the area of its base is 88 sq.cm. Find its height. (Ans: 5cm)
2. A cuboidal water tank is 6 m long, 5 m wide and 4.5 m deep. How many litres of water can it hold? (Hint, 1 cub. cm = 1000 lit, Ans. 135000 litres)
3. How many 3 metre cubes can be cut from a cuboid measuring 18m × 12m × 9m. (Ans: 72)
4. A new metal cube is obtained by melting three cubes with sides measuring 3 cm, 4 cm and 5 cm respectively. Find the length of the side and total surface area of the new cube supposing no waste. (Ans: 6cm; 216 sq.cm)

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Now, $V_1 = \frac{4}{3}\pi r^3 \Rightarrow V_1 = \frac{4}{3}\times\pi\times\sqrt{\frac{3}{2\pi}}\,a\times\sqrt{\frac{3}{2\pi}}\,a\times\sqrt{\frac{3}{2\pi}}\,a = \sqrt{\frac{6}{\pi}}\,a^3 = 1.38\ V_2$

Hence Proved.

Ex 4. How many solid circular cylinders, each of length 8cm and diameter 6 cm can be made out of the material of a solid sphere of radius 6 cm.

Solution: Volume of a sphere: $V = \frac{4}{3}\pi r^3$, radius: $r = 6$ cm

$\Rightarrow V = \frac{4}{3} \times \pi \times 6 \times 6 \times 6$ cub.cm

Again, Volume of a circular cylinder: $V_1 = \pi r^2 h$,

Given, diameter $= 6$ cm $\Rightarrow r = 3$cm, $h = 8$ cm

$\therefore$ Volume of each circular cylinder, $V_1 = \pi \times 3 \times 3 \times 8$ cub. cm

$\therefore$ Required number of cylinders $= V \div V_1 = 4$

Ex 5. Find the volume and total surface area of a hemisphere of radius 3.5 cm.

Solution: Given, radius: $r = 3.5$ cm

$\therefore$ Volume of the hemisphere $= \frac{2}{3}\pi r^3 = \frac{2}{3}\times\frac{22}{7}\times 3.5\times 3.5\times 3.5$ cub. cm

$= 89.83$ cub. cm

And, Total Surface Area $= 3\pi r^2 = 3\times\frac{22}{7}\times 3.5\times 3.5$ sq.cm $= 115.5$ sq. cm

Ex 6. [2013] Find the volume and curved surface area of a sphere of radius 6 cm.

Solution: Volume of a sphere: $V = \frac{4}{3}\pi r^3$, radius: $r = 6$ cm

$\Rightarrow V = \frac{4}{3} \times \pi \times 6 \times 6 \times 6$ cub.cm $= 905.14$ cub. cm

Surface Area $= 4\pi r^2 = 4 \times \frac{22}{7} \times 6 \times 6 = 452.57$ sq. cm

Ex 7. [2009] How many spherical balls of 2.5 mm diameter each can be obtained by melting a semicircular disc of 7 cm diameter and 8 mm thickness.

Solution: Volume of the semi circular disc:

$V = \frac{1}{2}\pi r^2 h$, $r = \frac{7}{2}$ cm $= 3.5$ cm $= 35$mm, $h = 8$mm

$\Rightarrow V = \frac{1}{2}\times\frac{22}{7}\times 35\times 35\times 8$ cub.mm

Volume of each spherical ball: $V_1 = \frac{4}{3}\pi r^3$, $r = \frac{2.5}{2}$ mm

$= \frac{4}{3}\times\frac{22}{7}\times\frac{2.5}{2}\times\frac{2.5}{2}\times\frac{2.5}{2}$ cub.mm

Therefore, required number of balls $= V \div V_1 = 1881.6 \sim 1881$

Ex 8. [2010] Three spheres of radii 3 cm, 4 cm and 5 cm are melted to form a new sphere. Find the radius of the new sphere.

Solution: Let r_1, r_2, r_3 be the radii and V_1, V_2, V_3 be the volume of the three given spheres respectively, also let r be the radius and V be the volume of the new sphere. $\therefore$ $r_1 = 3$ cm, $r_2 = 4$ cm, $r_3 = 5$cm

and $V_1 = \frac{4}{3}\pi (r_1)^3 = \frac{4}{3}\times\pi\times(3)^3 = 36\pi$ cu cm

$V_2 = \frac{4}{3}\pi (r_2)^3 = \frac{4}{3}\times\pi\times(4)^3$ cu cm $= \frac{256}{3}\pi$cu cm

$V_3 = \frac{4}{3}\pi (r_3)^3 = \frac{4}{3}\times\pi\times(5)^3$ cu cm $= \frac{500}{3}\pi$cu cm

Now, $V = V_1+V_2+V_3 = (36+\frac{256}{3}+\frac{500}{3})\pi$cu cm $= 288\pi$cu cm

again, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{4}{3}\pi r^3 = 288\pi$cu cm $\Rightarrow r^3 = 216$ cub. cm $\Rightarrow r = 6$ cm

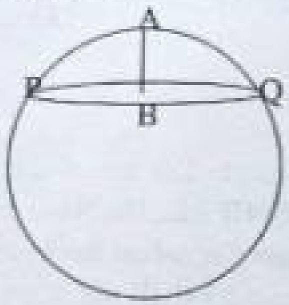
Ex 9. [2011, 2016] A cylindrical vessel has a base of diameter 10 cm. It is completely filled by emptying into it the contents of a hemispherical bowl also of diameter 10 cm. Find the height of the cylinder.

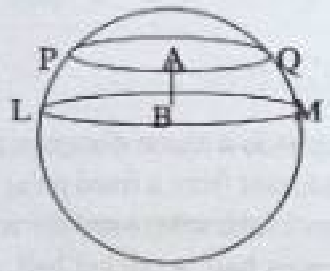
Solution: volume of the cylinder: $V_1 = \pi r^2 h$, $r = \frac{10}{2} = 5$ cm, h =?

$= \pi \times 5^2 \times h$ cub. cm

Volume of the hemispherical bowl: $V_2 = \frac{2}{3}\pi r^3$, $r = \frac{10}{2} = 5$ cm

$= \frac{2}{3}\times\pi\times 5^3$ cub. cm thickness AB. Zone PQML is defined by radius R of the sphere from which it is





cut off, r_1 of the circle PQ, r_2 of the circle LM, height h

Volume and Surface Area:

(a) For a sphere of radius 'r':

(i) Volume: $V = \frac{4}{3}\pi r^3$ cu units

(ii) Surface Area $= 4\pi r^2$ sq units

(b) For a Hemisphere of radius 'r':

(i) Volume: $V = \frac{2}{3}\pi r^3$ cu units

(ii) Curved Surface Area $= 2\pi r^2$ sq. units

(iii) Total Surface Area $= 2\pi r^2 + \pi r^2 = 3\pi r^2$ sq. units

(c) For a Spherical shell:

(i) Volume: $V = \frac{4}{3}\pi(R^3 - r^3)$ cu unit

(ii) Outer Surface Area $= \pi R^2$ sq unit

(iii) Inner Surface Area $= \pi r^2$ sq unit

(d) Volume of the segment $= \frac{1}{6}\pi h\,(3r^2 + h^2)$

Area of the curved surface of the segment $= 2\pi Rh$

Total surface area of the segment = Area of the curved surface + area of the base

$= 2\pi Rh + \pi r^2$

Volume of the zone $= \frac{1}{6}\pi h\,\{3(r_1^2 + r_2^2) + h^2\}$

Area of the curved surface of the zone $= 2\pi Rh$

Total surface area of the zone = Area of the curved surface + area of the two bases

$= 2\pi Rh + \pi r_1^2 + \pi r_2^2$

Worked out examples:

Ex 1. A solid sphere of radius 3 cm is melted and then cast into small spherical balls each of diameter 0.6 cm. Find the number of balls thus obtained.

Solution: Volume of the solid sphere: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi 3^3 = 360$ cub.cm

Radius of each spherical ball $= \frac{0.6}{2}$ cm $= 0.3$ cm

$\therefore$ Volume of a spherical ball: $V_1 = \frac{4}{3}\pi(0.3)^3$ cub.cm

$\therefore$ Required number of balls $= V \div V_1 = \frac{36\pi \times 3}{4 \times \pi \times 0.3 \times 0.3 \times 0.3} = 1000$

Ex 2. Show that the surface area of a sphere is same as that of the lateral surface of a right circular cylinder that just encloses the sphere.

Solution: Let 'r' be the radius of the sphere.

$\therefore$ Surface Area of the sphere $= 4\pi r^2$

A right circular cylinder just encloses the sphere

Now, for the cylinder, radius $= r$, height $= r + r = 2r$

$\therefore$ Lateral Surface Area of the cylinder

$= 2\pi r 2r = 4\pi r^2$. Hence Proved.

Ex 3. A sphere and a cube have the same surface area. Show that the volume of the sphere is 1.38 times that of the cube.

Solution: Surface Area of a Sphere: $S_1 = 4\pi r^2$

Surface Area of a cube: $S_2 = 6a^2$

Given, $S_1 = S_2 \Rightarrow 4\pi r^2 = 6a^2 \Rightarrow r^2 = \frac{3}{2\pi}a^2 \Rightarrow r = \sqrt{\frac{3}{2\pi}}\,a$

Volume of the Sphere: $V_1 = \frac{4}{3}\pi r^3$, Volume of the cube: $V_2 = a^3$

with $r_1 = 30$ cm, $r_2 = 30.5$ cm its volume $= \frac{22}{7} \times l\,(30.5^2 - 30^2)$

Given $11440000 = \frac{22}{7} \times l\,(30.5^2 - 30^2) = 95.07\,l \Rightarrow l = 120332$ cm = 1203.32 mt

Ex 11. [2013] Find the amount of concrete required to erect a concrete pillar whose circular base will have a perimeter 8.8 mt and whose curved surface area will be 17.6 sq.m.

Solution: Amount of concrete required = volume of the cylindrical pillar $= \pi r^2 h$

Given, $2\pi r = 8.8$ mt $\Rightarrow 2 \times \frac{22}{7} \times r = 8.8 \Rightarrow r = 1.4$ mt

and $2\pi rh = 17.6$ sq.m $\Rightarrow 8.8 \times h = 17.6 \Rightarrow h = 2$ mt

$\therefore$ Amount of concrete required $= \frac{22}{7} \times 1.4 \times 1.4 \times 2 = 12.32$ cub.m

Ex 12. [2017] A cylindrical oil tanker has its inner diameter 480 cm. If the length of the tanker is 6 mt, find the capacity of the tanker in cc.

Solution: Capacity = Volume of the tanker $= \frac{22}{7} \times 240^2 \times 600$, $r = 240$ cm, $l = 600$ cm

$= 10{,}86{,}17{,}142.85$ cu cm

Exercise 3.4

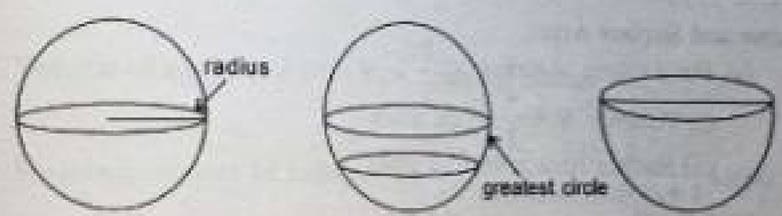
1. What is the ratio between the height and diameter of the base of a cylinder when the area of the curved surface is equal to sum of the areas of the two ends. (Ans. 1:2)

2. Find the weight of iron in a pipe whose interior and exterior diameters measure 10 inch and 11 inch respectively and length 10 ft, 1 cub. inch of iron weighing 0.26 lb. (Ans. 514.594 lb)

3. Find the amount of sand required to fill a cylindrical container having base radius 24 inch and height 5 ft. (Ans. 62.86 cub ft)

* * *

3.5 Sphere: It is a three dimensional solid object. The set of all points in space equidistant from a fixed point is called a SPHERE. The fixed point is called the *centre* of the sphere and the constant distance is called the *radius* of the sphere. A tennis ball, a cricket ball, a fully blown football are some Examples of a sphere. A sphere can also be obtained by rotating a circle about its diameter.



Sphere Section by a plabe Hemisphere

A section of a sphere by a plane is a circle. The plane through the centre of a sphere gives the greatest circle.

HEMISPHERE: A plane through the centre of a sphere divides the sphere into two parts, each of these parts is called a Hemisphere. The plane end is called its base.

SPHERICAL SHELL: The difference of two solid concentric spheres is called a Spherical Shell. A rubber ball made of a thin rubber sheet can be considered as the difference of two solid spheres which is a spherical shell.

Segment and Zone of Sphere:

AB is the height of the segment PAQ in fig 4.1. PQ is a circle formed by cutting off the sphere of radius R by a plane. PAQ segment is defined by its height h, radius r of the circle PQ, radius R of the sphere from which it is cut off. In figure AB is height of the zone PQML. PQ, LM are two parallel circles of

Solution: For the cylinder, $h = 14$ cm, radius $= r$ cm

$\therefore$ Volume of the cylinder: $V_1 = \pi r^2 h = 3.1416 \times r^2 \times 14$ cub. cm

For the Cube, side $= a = 11$cm

$\therefore$ Volume of the Cube: $V_2 = a^3 = 11 \times 11 \times 11$ cub. cm $= 1331$ cub. cm

According to question, $V_1 = V_2 \Rightarrow 3.1416 \times r^2 \times 14 = 1331 \Rightarrow r^2 = 30.26 \Rightarrow r = 5.5$

$\therefore$ Required radius $= 5.5$ cm

Ex 4. The whole surface of a right circular cylinder is 1540 sq. cm and the diameter of the base is half the height. Find the height of the cylinder.

Solution. Given, whole surface of the cylinder

$= 1540$ sq. cm and $d = \frac{h}{2} \Rightarrow h = 2d \Rightarrow h = 4r$

$\Rightarrow 2\pi r(r + h) = 1540$ sq. cm

$\Rightarrow 2\pi r(r + 4r) = 1540$ sq.cm

$\Rightarrow 10\pi r^2 = 1540$ sq. cm $\Rightarrow r^2 = \frac{1540 \times 7}{10 \times 22}$ sq. cm

$\Rightarrow r = 7$cm, therefore, $h = 4 \times 7$cm $= 28$cm

Ex 5. The volume of a metallic cylindrical pipe is 748 cub. cm. Its length is 14 cm and its external radius is 9 cm. Find its thickness.

Solution: A metallic cylindrical pipe resembles a hollow cylinder.

Now, Volume of a hollow cylinder: $V = \pi(R^2 - r^2)\,h$

Given, $V = 748$ cub.cm, $R = 9$ cm, $h = 14$ cm.

$\therefore \frac{22}{7} \times [\,(9)^2 - r^2] \times 14 = 748$

$\Rightarrow 81 - r^2 = \frac{748 \times 7}{22 \times 14} \Rightarrow r^2 = 81 - 17 \Rightarrow r^2 = 64 \Rightarrow r = 8$ cm.

Therefore, required thickness of the pipe $= R - r = (9 - 8)$ cm $= 1$cm.

Ex 6. [2009, 2010] Find the volume and curved surface of a right circular cylinder whose height is 12 cm and diameter of the base 3 cm.

Solution: Given, $h = 12$cm, diameter $= 3$ cm, therefore, radius $r = \frac{3}{2}$cm

$\therefore$ Required Volume of the cylinder $= \pi r^2 h = 3.1416 \times \frac{3}{2} \times \frac{3}{2} \times 12$ cu.cm

$= 84.86$ cub.cm

$\therefore$ Required Curved Surface of the cylinder $= 2\pi rh = 2 \times 3.1416 \times \frac{3}{2} \times 12$ sq.cm

$= 113.14$ sq.cm

Ex 7. [2010, 2015] Find the volume and whole surface area of a right circular cylinder whose height is 14cm and diameter of the base circle is 10cm.

Solution: Given, height: $h = 14$cm, diameter of the base $= 10$cm, radius: $r = 5$ cm.

$\therefore$ Volume of the cylinder $= \pi r^2 h = \frac{22}{7} \times 5 \times 5 \times 14$ cub.cm $= 1100$ cu.cm

And, whole surface area $= 2\pi r\,(h + r) = 2 \times \frac{22}{7} \times 5 \times (14 + 5)$sq. cm$= 597.14$ sq. cm

Ex 8. [2014] Find the volume and curved surface of a right circular cylinder whose height is 12 metre and diameter of the base 6 metre.

Solution: Given, $h = 12$ mt, diameter $= 6$ mt, therefore, radius $r = 3$ mt

Volume $= 3.14 \times 3^2 \times 12 = 339.4$ cu m, Curved Surface $= 2 \times 3.14 \times 3 \times 12 = 339.4$ sq mt

Ex 9. [2016] The area of the whole surface of a right circular cylinder is 3000 sq cm and diameter of the base is 40 cm. Find the volume and the height of the cylinder.

Solution: Area of whole surface of the cylinder of radius r and height h

$3000 = 2 \times \frac{22}{7} rh + 2 \times \frac{22}{7} \times r^2$

$= 2 \times \frac{22}{7} \times 20h + 2 \times \frac{22}{7} \times 20^2 = 125.7(20 + h)$

Hence $h = \frac{3000}{125.7} - 20 = 3.86$ cm

Volume $= \frac{22}{7} \times 20^2 \times 3.86$ cu cm $= 4860.55$ cu cm.

Ex 10. [2015] A solid iron rectangular block of dimensions 4.4 mt, 2.6 mt, 1 mt is cast into a hollow cylindrical pipe of internal radius 30 cm and thickness 5 cm. Determine the length of the pipe.

Solution: Volume of the iron block $= 440 \times 260 \times 100$ cu cm $= 11440000$ cu cm

If length of the pipe is l cm the pipe is of shape of a hollow cylinder

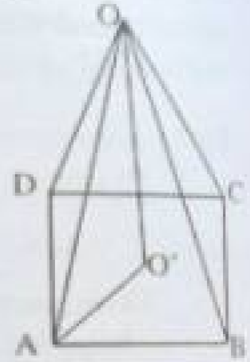
Solution: OABCD is a square pyramid. Given, each edge = 30 m.

Volume of the pyramid $= \frac{1}{3}$ Ah, A = (side)^2 = $(30)^2$ sq. m;

$$AO' = \frac{1}{2} AC = \frac{1}{2}\sqrt{AB^2 + BC^2} = \frac{1}{2}\sqrt{30^2 + 30^2} = 15\sqrt{2}$$

$$h = OO' = \sqrt{OA^2 - AO'^2} = \sqrt{(30)^2 - (15\sqrt{2})^2} = 15\sqrt{2}$$

Reqd volume$= \frac{1}{3} \times (30)^2 \times 15\sqrt{2}$ cub.m = 6363.96 cub.m



Ex 5. [2009] A right pyramid of height 12 cm stands on a square base whose side is 10 cm. Find the volume of the pyramid.

Solution: Volume of the Pyramid $= \frac{1}{3}$ Ah, A= Area of the square.

Given, h =12 cm, a =10 cm. ∴ A = $a^2 = (10)^2$ sq. cm =100 sq. cm

Therefore Required Volume $= \frac{1}{3} \times 100 \times 12$ cub. cm = 400 cub. cm

Ex 6. [2009] The area of the base of a hexagonal pyramid is $54\sqrt{3}$ and the area of one of its side faces is $9\sqrt{6}$. Find the volume of the pyramid.

Solution: OABCDEF is a hexagonal pyramid.

Volume of the pyramid

$= \frac{1}{3} \times$ Area of the base ×height

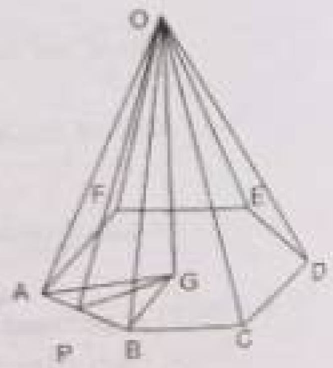
Given, Area of the base = $54\sqrt{3}$

$\Rightarrow \frac{3\sqrt{3}}{2} (a)^2 = 54\sqrt{3} \Rightarrow a = 6$ unit

Again, Area of OAB = $9\sqrt{6}$

$\Rightarrow \frac{1}{2} \times AB \times OP = 9\sqrt{6} \Rightarrow \frac{1}{2} \times 6 \times OP = 9\sqrt{6}$

$\Rightarrow OP = 3\sqrt{6}$ unit



$$\text{height} = h = OG = \sqrt{OP^2 - PG^2} = \sqrt{(3\sqrt{6})^2 - (\frac{\sqrt{3}}{2}AB)^2}, \text{ since } \Delta ABG \text{ is equilateral}$$

$$= \sqrt{(3\sqrt{6})^2 - (\frac{\sqrt{3}}{2} \times 6)^2} = 3\sqrt{3} \text{ unit}$$

∴ Required Volume $= \frac{1}{3} \times 54\sqrt{3} \times 3\sqrt{3}$ cub.unit = 162 cu unit.

Ex 7. [2014, 2015] The base of the pyramid is a square and its faces are equilateral triangles. If 'a' is the side of the base, show that its volume is $\frac{\sqrt{2}}{6} a^3$.

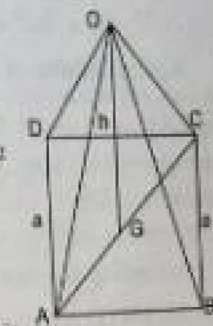
Solution: The pyramid is a square pyramid. Volume $= \frac{1}{3}$ Ah.

Here, A = a^2 sq. unit.

$$h = \sqrt{OA^2 - AG^2} = \sqrt{OA^2 - (\frac{1}{2}AC)^2}, \; AC^2 = AB^2 + BC^2 = 2a^2$$

$$= \sqrt{a^2 - (\frac{\sqrt{2}}{2}a)^2} = \frac{a}{\sqrt{2}}$$

∴ Required volume $= \frac{1}{3} \times a^2 \times \frac{a}{\sqrt{2}} = \frac{a^3}{3\sqrt{2}} = \frac{\sqrt{2}}{6} a^3$ cu unit.



Ex 8. The base of a pyramid is an equilateral triangle of side 15 cm. If the vertical height of the pyramid is 20 cm, find its volume and total surface.

Solution: Volume of the pyramid $= \frac{1}{3}$ Ah, h = OG = 20 cm

A = Area of an equilateral triangle $= \frac{\sqrt{3}}{4} a^2$, a = 15 cm

Therefore required volume $= \frac{1}{3} \times \frac{\sqrt{3}}{4} \times 15 \times 15 \times 20$ cub. cm

= 649.52 cub. cm.

Here, OAB, OAC, OBC and ABC are equilateral triangles of side 15cm.

A Pyramid is called a *Right Pyramid* if the axis is perpendicular to the base or the foot of the perpendicular coincides with the central point of the base. Otherwise, it is called an *Oblique Pyramid.* A pyramid is called a *Regular Pyramid* if the base is a regular figure.

A pyramid is named according to the shape of its base. For Example, *triangular pyramid, rectangular pyramid, square pyramid, pentagonal pyramid etc.* A triangular pyramid is also called a *tetrahedron. **All the edges of a Right Regular Tetrahedron are equal.***

Volume and Surface Area:

(a) For any Pyramid: Volume = $\frac{1}{3}$ Ah cub. unit, A = area of the base, h = height.

(b) For a Right Regular Pyramid:

(i) Volume = $\frac{1}{3}$Ah cub. unit, A = area of the base, h = height.

(ii) Lateral Surface Area = $\frac{1}{2}$ Ps sq. unit, P = perimeter of the base, s = slant height.

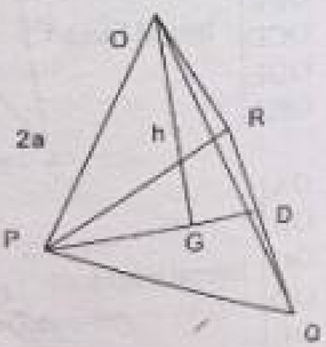
(iii) Total Surface Area = ($\frac{1}{2}$ Ps + A) sq. unit

Ex 1. Find the volume, lateral surface area and total surface area of a right regular tetrahedron of side '2a'.

Solution: Volume of a pyramid = $\frac{1}{3}$ Ah, A = area of the base, h = height.

Here, base is an equilateral triangle of side '2a' and all the edges are of length '2a', therefore all the faces are equilateral triangles.

$\therefore A = \frac{\sqrt{3}}{4}(2a)^2 = \sqrt{3}\,a^2$ sq.unit



$h = OG = \sqrt{OP^2 - PG^2}$

Now, PG = $\frac{2}{3}$ (median of PQR)

$= \frac{2}{3}PD = \frac{2}{3} \times \frac{\sqrt{3}}{2} \times 2a = \frac{2}{\sqrt{3}}a$

$\therefore h = \sqrt{(2a)^2 - (\frac{2}{\sqrt{3}}a)^2} = \frac{2\sqrt{2}}{\sqrt{3}}a$

$\therefore$ Required Volume = $\frac{1}{3} \times \sqrt{3}\,a^2 \times \frac{2\sqrt{2}}{\sqrt{3}}a$ cub. unit = $\frac{2\sqrt{2}}{3}a^3$ cub. unit.

Lateral surface area = 3A = $3\sqrt{3}\,a^2$ sq.unit

Total surface area = 4A = $4\sqrt{3}\,a^2$ sq.unit

Ex 2. A hexagonal pyramid has the perimeter of its base 15 metres and altitude is 10 metres. Find its volume.

Solution: Volume of a pyramid = $\frac{1}{3}$Ah, A = area of the base, h = height.

Given, perimeter of the base (hexagon) = 15 m, h = 10m.

$\Rightarrow$ side of the base = $\frac{15}{6}$ m = 2.5 m

$\Rightarrow A = \frac{3\sqrt{3}}{2}(\text{side})^2 = \frac{3\sqrt{3}}{2} \times (2.5)^2$ sq. m

Therefore, required volume = $\frac{1}{3} \times \frac{3\sqrt{3}}{2} \times (2.5)^2 \times 10$ cub. m = 54.13 cub. m.

Ex 3. [2011, 2015] The base of a pyramid is a regular hexagon of perimeter 42 cm. If the height of the pyramid is 26 cm, find its volume.

Solution: Given, a regular hexagonal pyramid.

Perimeter of the hexagon = 42 cm. Height = h = 26 cm.

$\therefore$ side of the hexagon = a = $\frac{42}{6}$ =7 cm.

Area of the hexagon = A = $\frac{3\sqrt{3}}{2}a^2 = \frac{3\sqrt{3}}{2}(7)^2$ sq. cm

$\therefore$ Required volume = $\frac{1}{3}$ Ah = $\frac{1}{3} \times \frac{3\sqrt{3}}{2}(7)^2 \times 26$ cub. cm = 1103.32 cub cm.

Ex 4. [2010] A pyramid on a square base has four equilateral triangles as its four faces, each edge being 30 meters. Find the volume of the pyramid.

Volume of the enclosed hemisphere and cone $=\frac{2}{3}\times\frac{22}{7}\times 5^3+\frac{1}{3}\times\frac{22}{7}\times 5^2\times 5$

$$=\frac{22}{7}\times 5^3$$

Volume of the additional enclosed space $=\frac{22}{7}\times 5^2\times 10-\frac{22}{7}\times 5^3=\frac{22}{7}\times 5^3$

Ex 16. Marbles of diameter1.4 cm are dropped into a beaker containing some water and are fully submerged. The diameter of the beaker is 7 cm. Find no of marbles to be dropped to raise the water by 5.6 cm.

Solution: The change in volume of water in the beaker is $V=\frac{22}{7}\times(\frac{7}{2})^2\times 5.6$

This volume is same as the total of volume of the immersed marble of radius .7cm.

Hence $V=n\times\frac{4}{3}\times\frac{22}{7}\times(.7)^3$ where n is total number of marbles.

Thus $\frac{22}{7}\times(\frac{7}{2})^2\times 5.6=n\times\frac{4}{3}\times\frac{22}{7}\times(.7)^3 \Rightarrow 2=n\times\frac{4}{300} \quad \Rightarrow n=150$

Ex 17. The radii of the external and internal surfaces of a spherical shell are 5 cm and 3 cm respectively. If the material were formed into a solid cylinder of height $10\frac{2}{3}$ cm what would be its diameter?

Solution: The volume of the spherical shell $=\frac{4}{3}\times\frac{22}{7}\,(5^3-3^3)$

Volume of the new cylinder $=\frac{22}{7}\times r^2\times 10\frac{2}{3}$

$$\Rightarrow \frac{4}{3}\times\frac{22}{7}\,(5^3-3^3)=\frac{22}{7}\times r^2\times 10\frac{2}{3}$$

$$\Rightarrow \frac{4}{3}\times 98=r^2\times\frac{32}{3} \qquad \Rightarrow r=\frac{7}{2}\text{ cm or diameter} = 7\text{ cm.}$$

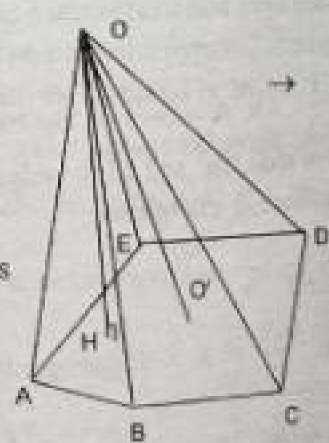
Exercise 3.5

1. Three solid spheres of iron with diameters 2 cm, 12 cm and 16 cm respectively are melted to form a single solid sphere. Find the radius of this sphere. (Ans. 9 cm)
2. How many spherical bullets can be made out of a solid cube of lead whose edge measures 44 cm, each bullet being 4 cm in diameter. (Ans. 2541)
3. Find the volume of the largest sphere that can be curved out of a cube of side 7 cm. (Hint. radius of the sphere $=\frac{7}{2}$ cm. (Ans. 179.67 cub.cm)
4. How many lead balls of quarter of an inch in diameter can be cast out of a ball of 3 inch in diameter supposing no waste. (Ans. 1728)

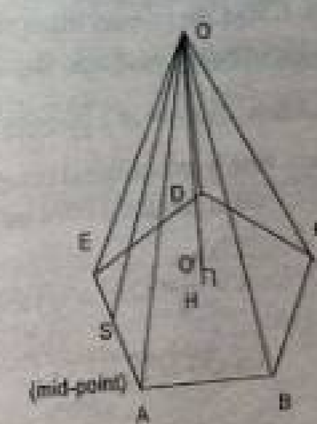
* * *

3.6 Pyramid: A Pyramid is a solid whose base is a polygon of any number of sides, with triangular faces meeting at a common point. We can say that a Pyramid can be obtained by joining the vertices of a polygon to a point not lying in the plane of the polygon.

O → vertex
OO' → axis
OH → height
ABCDE → base
OS → slant height
OAB, OBC, OCD, ODE, OEA — lateral faces
OA, OB, OC, OD, OE — lateral edges



Oblique Pyramid



Right Pyramid

According to question, $V_1 = V_2 \Rightarrow h = \frac{10}{3}$ cm = 3.33 cm

Ex 10. [2012] Find the area of the curved surface of the segment of a sphere of height 12 inch and radius 20 inch.

Solution: Curved surface area of the segment of a sphere = $2\pi rh$

Here, r = 20 inch, h = 12 inch,

$\therefore$ Required curved suface area

$$= 2\times\frac{22}{7}\times 20\times 12 \text{ sq.inch} = \text{sq.inch} = 1508.57 \text{ sq. inch.}$$

Ex 11. The radius of a sphere is 13 cm. It is cut by a plane whose perpendicular distance from the centre of the sphere is 5 cm. Calculate the circumference of the plane circular section. Also find volume of the smaller segment.

Solution: Radius of the circular section formed by intersection of the plane with the sphere is $\sqrt{13^2-5^2} = 12$.

$$\text{Circumference of the circular section} = 2\times\frac{22}{7}\times 12 = 77.43 \text{ cm.}$$

$$\text{Volume of the smaller segment} = \frac{1}{6}\times\frac{22}{7}\times 8\,(3\times 12^2 + 8^2) \qquad h = 13-5 = 8$$

$$= 2078.4 \text{ cu cm}$$

Ex 12. Find the volume and curved surface of a zone of a sphere, the radii of the two ends being 20 cm and 10 cm and its height is 7 cm.

Solution: Volume of the zone $= \frac{1}{6}\times\frac{22}{7}\times 7\,[3(20^2 + 10^2) + 7^2]$

$$= 5679.67 \text{ cu cm}$$

Radius of the sphere is R. We assume d as the distance of larger base of the zone from the centre of the sphere. So distance of smaller base from the centre is $d+7$. Hence $R^2 = d^2 + 20^2 = (d+7)^2 + 10^2$

$\Rightarrow 400 = 14d + 149 \Rightarrow d = 251/14 = 17.9 \Rightarrow R = \sqrt{17.9^2+20^2} = 26.9$ cm

Area of the curved surface of the zone $= 2\times\frac{22}{7}\times 26.9\times 7 = 1181.9$ sq cm.

Ex 13. An evaporating pan is in the shape of a segment of a sphere. Its rim diameter is .6 m and is 27.5 cm deep. Calculate the internal curved surface area.

Solution: Radius of the sphere is R. We assume d as the distance of base of the segment from the centre of the sphere. Hence $d = R - 27.5$. Radius of the rim is 30 cm. And $R^2 = d^2 + 30^2 = (R-27.5)^2 + 30^2$

$$\Rightarrow R^2 = R^2 - 55R + 1656.25 \quad \Rightarrow R = 1656.25/55$$

Internal curved surface area $= 2\times\frac{22}{7}\times 1656.25/55\times 27.5 = 5205.35$ sq m.

Ex 14. The radius of the base of a metallic spherical segment is 20 cm and height is 12 cm. If the segment is hammered out into a circular plate of 40 cm diameter, find the thickness of the plate.

Solution: Volume of the spherical segment $= \frac{1}{6}\times\frac{22}{7}\times 12\,(3\times 20^2 + 12^2)$

If h is the thickness or height of the cylinder then

$$\text{Volume of the cylinder} = \frac{22}{7}\times 20^2 h$$

Given Volume of the spherical segment = Volume of the cylinder

Or, $\frac{1}{6}\times\frac{22}{7}\times 12\,(3\times 20^2 + 12^2) = \frac{22}{7}\times 20^2 h$

$$\Rightarrow h = \frac{2\times 1344}{20^2} = 6.72\text{cm.}$$

Ex 15. A right cone and a hemisphere lie on opposite sides on a common base 10 cm in diameter. The cone is right angled at the vertex. If a cylinder circumscribes them in this position what additional space will be enclosed?

Solution: The cone is right angled means height h of the cone is half the diameter of the base, i.e. $h = 5$ cm. Hemisphere is of radius 5 cm. So height of the circumscribing cylinder is of height $5+5 = 10$ cm and is of radius 5 cm.

$$\text{Volume of the cylinder} = \frac{22}{7}\times 5^2\times 10$$

$$\text{Volume of the tent} = 2355 = \frac{1}{3}\times 314 \times h \text{ cu cm}$$

$$\Rightarrow \text{Height of the cone} = h = \frac{3\times 2355}{314} = 22.5 \text{ mt.}$$

$$\text{Slant height} = \sqrt{(22.5)^2 + (9.99)^2} = 24.6 \text{ mt.}$$

Ex 13. [2016] A circular tent is cylindrical to a height of 4 mt and conical above it. If its diameter is 105 mt and its slant height is 40 mt, calculate the total area of canvas required. What will be the cost of canvas at Rs 50 per meter if it is of width 1.5 mt?

Soln: Curved surface area of the cylindrical part

$$= \frac{22}{7}\times 105 \times 4 \text{ sq mt} = 1320 \text{ sq mt}$$

Curved surface area of the conical part

$$= \frac{1}{2}\times\frac{22}{7}\times 105 \times 40 \text{ sq mt} = 6600 \text{ sq mt}$$

Total surface area = 6600 + 1320 sq mt = 7920 sq mt

= Area of the canvas required

Length of the canvas = 7920/(1.5) mt = 5280 mt

Cost = Rs 50 × 5280 = Rs 264000

Ex 14. [2016] Water flows at the rate of 10 mt per min from a cylindrical pipe 5 mm in diameter. How long would it take to fill a conical vessel whose base diameter is 20 mt and depth is 24 mt?

$$\text{Soln: Volume of the vessel} = \frac{1}{3}\times\frac{22}{7}\times 10^2 \times 24 \text{ cu mt} = 2514.2 \text{ cu mt}$$

$$\text{Water flows in 1 min} = \frac{22}{7}(2.5\times 10^{-3})^2 \times 10 \text{ cu mt}$$

Hence time taken to fill the vessel is

Volume of the vessel/ Water flows in 1 min

$$= \frac{10^8 \times 24}{3\times 2.5\times 2.5\times 10} = 1.28\times 10^7 \text{ min} = \frac{10^7\times 1.28}{60} \text{ hr} = .2133\times 10^6 \text{ hr}$$

Exercise 3.7

1. A spherical cannon ball 6 inch in diameter is melted and cast out into a conical mould. If the diameter of the base is 12 inch, find the height of the cone. (Ans. 3 inch)
2. What is the length of a 5m wide canvas required to make a conical tent 24m high with base radius 7m. (Ans. 110m)
3. The radius of a cone is 7 cm and area of the curved surface is 176 sq. cm. Find the slant height of the cone. (Ans. 8 cm)
4. Find the volume of the right circular cone that can be cut out of a cube of edge 3 ft. (Ans.7.0686cub.ft)
5. The radius and height of a cone are in the ratio 3:4. If its volume is 301.44 cub. cm, find the radius and slant height. (Ans. radius = 6 cm, slant height = 10 cm).
6. Find the volume and curved surface area of a right circular cone whose slant height is 10 cm and semi vertical angle is 30°. (Ans. V = 226.8 cu cm, S = 157.1 sq cm

* * *

3.8 Frustum Of Right Circular Cone And Right Pyramid:

If a Right Circular Cone or a Right Pyramid is cut off by a plane parallel to its base, then the solid obtained is called Frustum of a Right Circular Cone or Frustum of a Right Pyramid. As shown in the above figures, the Frustum of a right circular cone is bounded by two circular planes and a curved surface. The Frustum of a right pyramid is bounded by two polygons at the two ends and its lateral faces are trapezoids.

The perpendicular distance between the two parallel ends is the *height* of the frustum. For the frustum of a cone, *slant height* is the distance between two parallel radii of the two ends along the surface of the cone. For the frustum of a pyramid, *slant height* is the line joining the mid points of two parallel edges of the two ends.

Canal,

$\therefore$ Area of the canvas used = 235.71 sq. m

Given that, width of the canvas used = 3m.

$\therefore$ Length of the canvas used $= \dfrac{area}{width} = \dfrac{235.71}{3}$ m =78.57 m

Ex 7. [2013, 2016] Find the volume and curved surface area of a right circular cone whose height is 6 cm and radius of the circular base is 2 cm.

Solution: Volume of a right circular cone : $V = \frac{1}{3} Ah,\ A = \pi r^2$

and Curved Surface Area $= \pi r \sqrt{h^2 + r^2}$, h = 6 cm, r = 2 cm.

$\therefore$ Required Volume, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times 3.1416 \times (2)^2 \times 6$ cub. cm = 25.12 cub. cm

$\therefore$ Curved Surface Area $= 3.1416 \times 2 \times \sqrt{6^2 + 2^2}$ sq. cm = 39.74 sq. cm

Ex 8. [2013]The volume of a right circular cylinder and a right circular cone standing on the same base are in the ratio 3:2.Show that the height of the cone is double the height of the cylinder.

Solution: volume of the right circular cylinder $= \pi r^2 h$

volume of the right circular cone $= \frac{1}{3}\pi r^2 h_1$

Given, $\pi r^2 h : \frac{1}{3}\pi r^2 h_1 = 3 : 2 \Rightarrow h : h_1 = 1 : 2 \Rightarrow h_1 = 2h$. Hence proved.

Ex 9. A solid cube of side 7 cm is melted to make a cone of height 5 cm, find the radius of the base of the cone.

Solution: Given, side of the cube = 7cm, height of the cone = h = 5cm,

Volume of the cube: $V_1 = (\text{side})^3 = 7 \times 7 \times 7$ cub. cm = 343 cub. cm

Volume of the cone: $V_2 = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times r^2 \times 5$ cub. cm

According to question, $V_1 = V_2 \Rightarrow \frac{1}{3} \times \frac{22}{7} \times r^2 \times 5 = 343$

$$\Rightarrow r^2 = \frac{343 \times 3 \times 7}{22 \times 5} \Rightarrow r^2 = 65.4818 \Rightarrow r = 8.09 \text{ cm}$$

Ex 10. [2013] A tent is in the form of a right circular cylinder surmounted by a cone. The diameter of the cylinder is 24 ft.The height of the cylindrical portion is 11 ft while the vertex of the cone is 16 ft above the ground. Find the area of the canvass required for the tent.

Solution: For the cylinder,

Curved surface area $= 2\pi rh$, $r = \frac{24}{2} = 12$ft, h = 11ft

$= 2 \times \frac{22}{7} \times 12 \times 11 = 829.71$ sq.ft

For the cone: Curved surface area $= \pi rs$, r = 12ft, h = 16 - 11 = 5ft

$s = \sqrt{r^2 + h^2} = \sqrt{12^2 + 5^2} = 13$ft

$\therefore$ Curved surface area $= \frac{22}{7} \times 12 \times 13 = 490.29$sq.ft

$\therefore$ Canvass required = 829.71 sq.ft + 490.29sq.ft.=1320 sq. ft.

Ex 11. [2014] The section of a right circular cone by a plane through its vertex perpendicular to the base is an equilateral triangle each side of which is 12 mt. Find the volume of the cone.

Soln: The diameter of the circular base of the cone is 12, also its slant height is 12

Hence height of the cone $= \sqrt{12^2 - 6^2}$ where 6 is radius

= 10.4 mt.

Volume $= \frac{1}{3} \times \frac{22}{7} \times 6^2 \times 10.4$ cu mt = 392.2 cu mt

Ex 12. [2015] A conical tent is required to accommodate 157 persons, each person must have 2 sq mt of space on the ground and 15 cu mt of air to breadth. Find the height of the tent and also calculate slant height.

Soln: Volume of the tent = 157 × 15 cu mt = 2355 cu mt

Area of the base = 157 × 2 sq mt = 314 sq mt

$= \frac{22}{7} \times r^2 \quad \Rightarrow$ Radius of the base $= r = \sqrt{\frac{7 \times 314}{22}} = 9.99$ mt

Volume and Surface Area:

(a) For a Circular Cone: Volume $=\frac{1}{3}\times$Area of the base $\times$ height $=\frac{1}{3}\pi r^2h$ cu unit

(b) For a Right Circular Cone:

(i) Volume $=\frac{1}{3}\times$ Area of the base $\times$ height $=\frac{1}{3}\pi r^2h$ cu unit

(ii) Lateral Surface Area $=\pi rs$ sq. unit; $s=\sqrt{h^2+r^2}$

(iii) Total Surface Area $=\pi r(s+r)$ sq. unit; $s=\sqrt{h^2+r^2}$

Worked out examples:

Ex 1. The diameter of a right circular cone is 14 cm and its slant height is 9 cm. Find the volume and area of the curved surface.

Solution: For a right circular cone, volume $=\frac{1}{3}\pi r^2h$, area of the curved surface $=\pi rs$.

Given, diameter = 14 cm, s = 9 cm.

$\therefore r=\frac{14}{2}=7$ cm, $h=\sqrt{s^2-r^2}=\sqrt{9^2-7^2}=4\sqrt{2}$ cm.

$\therefore$ Required volume $=\frac{1}{3}\times\frac{22}{7}\times 7\times 7\times 4\sqrt{2}$ cub. cm=290.39 cub. cm

$\therefore$ Required curved surface $=\frac{22}{7}\times 7\times 9$sq. cm = 198 sq. cm

Ex 2. [2009, 2017] Find the volume and curved surface area of a right circular cone whose height is 8 cm and radius of the circular base is 3 cm.

Solution: Volume of a right circular cone : $V=\frac{1}{3}Ah$, $A=\pi r^2$

and Curved Surface Area $=\pi r\sqrt{h^2+r^2}$, h = 8 cm, r = 3 cm.

$\therefore$ Required Volume, $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\times 3.1416\times(3)^2\times 8$ cub. cm

= 75.40 cub. cm

$\therefore$ Curved Surface Area $=3.1416\times 3\times\sqrt{8^2+3^2}$ sq. cm = 80.53 sq. cm

Ex 3. [2011] How many square metres of canvas is used in a conical tent whose height is 35 metres and radius of the base 84 metres?

Solution: Height of the cone: h = 35m, radius of the base: r = 84m.

slant height: $s=\sqrt{h^2+r^2}=\sqrt{35^2+84^2}=91$m

Required length of canvas = Curved surface area of the cone $=\pi rs$

$=\frac{22}{7}\times 84\times 91$ sq. m = 24024 sq. m

Ex 4. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 9 cm.

Solution: The base of the cone is the circle inscribed in the base of the cube and the height is equal to the length of an edge of the cube.

Volume of the cone $=\frac{1}{3}\pi r^2h$, here, $r=\frac{9}{2}$ cm, h = 9 cm

Therefore, required volume $=\frac{1}{3}\times\frac{22}{7}\times\frac{9}{2}\times\frac{9}{2}\times 9$ cu cm

= 190.93 cub. cm

Ex 5. The area of the whole surface of a right circular cone is 15 sq. m and the slant height is three times the radius of the base. Find the radius of the base.

Solution: Whole surface area of a right circular cone; $S=\pi r(r+s)$.

Given, S = 15 sq. m and s = 3r

$\therefore\ \frac{22}{7}\times r(r+3r)=15\Rightarrow 88r^2=105\Rightarrow r^2=1.1932\Rightarrow r=1.09$ m

Ex 6. How many metres of canvas 3m wide will be required to make a conical tent 11 metres high and 12 metres at the base ?

Solution: Canvas required = Lateral surface area of the cone $=\pi rs$, r = radius, s = slant height

Given, diameter of the base = 12 m, height: h = 11m

$\therefore r=\frac{12}{2}$ m = 6 m, $s=\sqrt{r^2+h^2}=\sqrt{6^2+11^2}=12.53$ m

$\therefore$ Lateral surface area of the cone $=\frac{22}{7}\times 6\times 12.5$sq. m = 235.71 sq. m

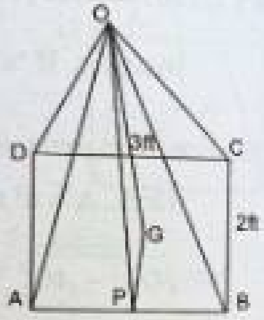
$\therefore$ Total surface area $= 4\times\frac{\sqrt{3}}{4}a^2 = \sqrt{3}\times15\times15$ sq. cm $= 389.71$ sq. cm.

Ex 9. The base of a pyramid is a rectangle measuring 2 ft by 3 ft and the slant height from vertex to either one of the longer sides is 5 ft. Find the height of a cylinder of base radius 6 inch, whose solid content is half that of the pyramid.

Solution. Volume of the rectangular pyramid: $V_1 = \frac{1}{3}$ Ah, A = length × breadth

Given, length = 3 ft, breadth = 2 ft, OP = 5 ft.

Therefore, $h = OG = \sqrt{OP^2 - PG^2} = \sqrt{5^2 - 1^2} = 2\sqrt{6}$ ft



$\therefore V_1 = \frac{1}{3}\times 3\times2\times2\sqrt{6}$ cub.ft $= 4\sqrt{6}$ cub. ft.

Again, Volume of the cylinder:

$V_2 = \pi r^2 h$, r = 6 inch $= \frac{1}{2}$ ft

$= \pi(\frac{1}{2})^2 h$ cub. ft

According to question, $\frac{1}{2}V_1 = V_2 \Rightarrow \frac{1}{2}4\sqrt{6} = \pi(\frac{1}{2})^2 h \Rightarrow h = 6.24$ ft

Ex 10. [2012] A right pyramid 10 ft high has a square base with diagonal 12ft. Find the volume.

Solution: Volume of the pyramid $= \frac{1}{3}$ Ah, $A = \frac{1}{2}d^2h$, d = 12 ft, h = 10 ft.

$\Rightarrow$ Required volume $= \frac{1}{3}\times\frac{1}{2}\times12\times12\times10 = 240$ cub.ft

Ex 11. [2016] Determine the volume of the pyramid whose height is $10\sqrt{7}$ ft and which stands on a triangle of sides 16 ft, 11 ft and 9 ft.

Solution: Area of the base of the pyramid

= Area of the triangle $= \sqrt{18(18-16)(18-9)(18-11)}$, $2s = 16+11+9 = 36$

$= \sqrt{18\times2\times9\times7} = 18\sqrt{7}$ sq ft

Volume of the pyramid $= \frac{1}{3}\times18\sqrt{7}\times10\sqrt{7}$ cu ft $= 420$ cu ft.

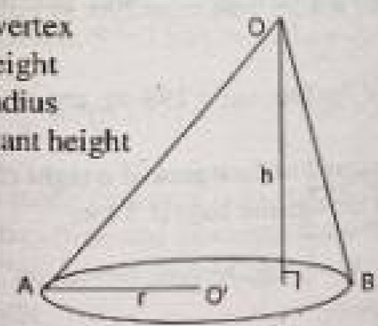
Exercise 3.6

1. A right pyramid 10 ft high has a square base of which the diagonal is 10 ft. Find the volume of the pyramid. (Ans. 166.67 cub ft)
2. The area of the base of a hexagonal pyramid is $284\sqrt{3}$ and the area of one of its side faces is $64\sqrt{6}$. Find the volume of the pyramid. (Ans. 3072 cub. unit)
3. A pyramid on a square base has every edge 100 ft. Find the edge of a cube of equal volume. (Ans. 61.7 cub. ft)
4. The perimeter of the base of a hexagonal pyramid is 15 metres, its altitude is 15 metres, find the volume. (Ans. 81.2 cub. m)
5. Find the volume and lateral surface area of a regular hexagonal pyramid of side 'a'. (Ans. $\frac{1}{6\sqrt{2}}a^3$, $\frac{3\sqrt{3}}{4}a^2$)

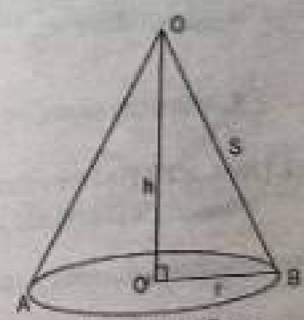
* * *

3.7 Cone: A Cone is a three dimensional solid bounded by a curved surface and a circular plane surface called its base. Its base being a circle, a cone is also known as a *Circular Cone*. The ice-cream cone, a circus clown's cap, a conical tent are some Ex.s of cone. A Cone can also be considered as a special case of a pyramid. If we go on increasing the number of sides in the base of a pyramid, at one time, it will resemble a circle and the pyramid will change into a cone.

O → vertex
h → height
r → radius
s → slant height



Circular Cone



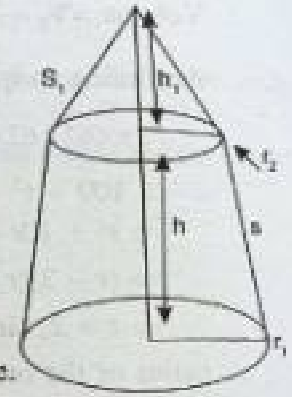
Right Circular Cone

A *Right Circular Cone* is generated by revolving a right triangle about one of its sides. The difference between a circular cone and a right circular cone is that, *ina right circular cone, the line joining the vertex to the centre of the baseisalways perpendicular to the base.*

$$= \frac{89914}{108} \text{ cub.ft}$$
$$= 832.54 \text{ cub.ft}$$

Ex 13. A tent is made in the form of a conic frustum surmounted by a cone. The diameters at the base and top of the frustum are 20 m and 6 m respectively and the height is 24 m. If the height of the tent is 28 m, find the quantity of canvas required to make the tent.

Solution: Let h be the height, s be the slant height of the frustum and r_1, r_2 be the radii of the circular ends.



Here, $h = 24$ m, $r_1 = \frac{20}{2}$ m $= 10$ m, $r_2 = \frac{6}{2}$ m $= 3$ m

$$s = \sqrt{h^2 + (r_1 - r_2)^2} = \sqrt{24^2 + (10-3)^2} = 25 \text{ m}$$

$\therefore$ Curved surface area of the frustum: $S_1 = \pi(r_1 + r_2)s$

$$= \frac{22}{7}(10+3) \times 25 \text{ sq. m}$$
$$= 1021.43 \text{ sq. m}$$

Let s_1 be the slant height and h_1 be the height of the cone.

$\therefore$ $h_1 = 28\text{-}24$ m $= 4$ m, $r_2 = 3$ m,

$$\Rightarrow s_1 = \sqrt{{h_1}^2 + {r_2}^2} = \sqrt{4^2 + 3^2} = 5 \text{ m}$$

$\therefore$ Curved surface area of the cone: $S_2 = \pi r_2 s_1 = \frac{22}{7} \times 3 \times 5$ sq. m $= 47.14$ sq. m

$\therefore$ Canvas required $= S_1 + S_2 = (1021.43 + 47.14)$ sq.m. $= 1068.57$ sq. m.

Ex 14. [2015, 2016] The perimeters of the ends of a frustum of a circular cone are 44 cm and 88 cm. If the height of the frustum is 16 cm, find its volume.

Soln: For one base perimeter $= 44 = 2 \times \frac{22}{7} \times r_1 \Rightarrow r_1 = 7$

For the other base perimeter $= 88 = 2 \times \frac{22}{7} \times r_2 \Rightarrow r_2 = 14$

$$\text{Volume} = \frac{1}{3} \times h\,(A_1 + A_2 + \sqrt{A_1 A_2}) \text{ cu cm}$$

$$= \frac{1}{3} \times 16\{\frac{22}{7}(7^2 + 14^2 + 7 \times 14)\} \text{ cu cm}$$
$$= \frac{1}{3} \times \frac{22}{7} \times 16 \times 343 \text{ cu cm}$$
$$= 5749.3 \text{ cu cm.}$$

Ex 15. [2014] Find the volume of the frustum of a pyramid 61 ft high, 7 ft 8 inches square at the base and 4 ft 6 inches at the top.

Soln: A_1 = area of the square base with side 7 ft 8 inches

$$= (7 \text{ ft } 8 \text{ inches})^2 = (7\tfrac{2}{3})^2 \text{ sq ft} = 58.7 \text{ sq ft.}$$

A_2 = area of the square base with side 4 ft 6 inches

$$= (4 \text{ ft } 6 \text{ inches})^2 = (4\tfrac{1}{2})^2 \text{ sq ft} = 20.25 \text{ sq ft.}$$

$h = 61$ ft

$$\text{Volume} = \frac{1}{3} \times h\,(A_1 + A_2 + \sqrt{A_1 A_2}) \text{ cu ft}$$
$$= \frac{1}{3} \times 61(58.7 + 20.25 + \sqrt{(\tfrac{23}{3})^2 (\tfrac{9}{2})^2}) \text{ cu ft}$$
$$= \frac{1}{3} \times 61(78.95 + \frac{23}{3} \times \frac{9}{2}) \text{ cu ft}$$
$$= \frac{1}{3} \times 61 \times 113.45 = 2306.8 \text{ cu ft.}$$

Exercise 3.8

1. Find the volume of the frustum of a square pyramid 61 ft high, 7 ft 8 inch square at the base, 4 ft 6 inch square at the top and capped by a square pyramid, if its height from the base to the apex is 7 ft 6 inch. (Ans. 2359.03 cub ft)

Solution: Area of the base of the tank: $A_1 = 40 \times 8$ sq. m = 320 sq. m,

Area of the top of the tank: $A_2 = 30 \times 6$ sq. m = 180 sq. m,

$$\text{Volume of the tank} = \frac{h}{3}\left[A_1 + A_2 + \sqrt{A_1 A_2}\right], \quad h = 3\text{ m}$$

$$= \frac{3}{3}\left[320 + 180 + \sqrt{320 \times 180}\right] \text{ cub. m}$$

$= 740$ cub. m

$= 740 \times 1000$ litre, 1 cub. m = 1000 lit.

$= 740000$ litre.

Ex 9. Find the volume of the frustum of a right circular cone, the radii of whose ends are 5 m and 8 m and the slant height is 4 m.

Solution: Let r_1, r_2 be the radii of the ends and s and h be the slant height and the height of the frustum respectively.

Given, $r_1 = 8$ m, $r_2 = 5$ m, $s = 4$ m.

$$\therefore h = \sqrt{s^2 - (r_1 - r_2)^2} = \sqrt{4^2 - (8-5)^2} = \sqrt{7}\text{ m}$$

$$\therefore \text{Required volume} = \frac{h}{3}\pi\,[r_1^2 + r_2^2 + r_1 r_2]$$

$$= \frac{\sqrt{7}}{3} \times \frac{22}{7}\,[8^2 + 5^2 + 8 \times 5]\text{ cub. m}, \sqrt{7} = 2.6458$$

$= 357.55$ cub. m.

Ex 10. [2011] Find the amount of water a bucket in the form of a frustum of a cone can hold if the radii of the top and bottom bases are 22 cm and 14 cm and the height of the bucket is 12 cm.

Solution: Given, $r_1 = 22$ cm, $r_2 = 14$ cm, $h = 12$ cm.

Required amount of water = Volume of the bucket

$$= \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 12 \times [(22)^2 + (14)^2 + 22 \times 14]\text{ cub. cm}$$

$$= \frac{88}{7} \times [484 + 196 + 308]\text{ cub. cm}$$

$= 12420.57$ cub.m

Ex 11. [2012] The volume of a conical frustum and that of a cylinder is same. The height of the cylinder is one third of a meter and radius is 62m. The height of the frustum is 31m and the radius of one end is 10m. Find the radius of the other end.

Solution: For the cylinder, $h = \frac{1}{3}$ m, $r = 62$m

$$\therefore \text{Volume} = V_1 = \pi r^2 h = \pi \times 62^2 \times \frac{1}{3}\text{ cub.m}$$

For the conical frustum, $h = 31$m, $R = 10$m

$$\therefore \text{Volume} = V_2 = \frac{\pi h}{3}[R^2 + r^2 + Rr] = \frac{\pi \times 31}{3}[10^2 + r^2 + 10r]\text{ cub.m}$$

According to question, $V_1 = V_2$

$$\Rightarrow \pi \times 62 \times 62 \times \frac{1}{3} = \frac{\pi \times 31}{3}[10^2 + r^2 + 10r]$$

$$\Rightarrow 100 + r^2 + 10r = 124$$

$$\Rightarrow r^2 + 10r - 24 = 0$$

$$\Rightarrow (r-2)(r+12) = 0$$

$$\Rightarrow r = 2, \text{ since } r \neq -12$$

$\therefore$ radius of the other end = 2m.

Ex 12. [2012] Find the cubical content of the frustum of a square pyramid 22 ft high, 7 ft 8 inch square at the base and 4 ft 6 inch square at the top.

Solution: Volume of the frustum of a pyramid: $V = \frac{h}{3}[A_1 + A_2 + \sqrt{A_1 A_2}]$

Here, $A_1 = a^2$, $a = 7$ ft 8 inch $= 7\frac{8}{12}$ ft $= 7\frac{2}{3}$ ft $= \frac{23}{3}$ ft

$A_2 = b^2$, $b = 4$ ft 6 inch $= 4\frac{6}{12} = 4\frac{1}{2} = \frac{9}{2}$ ft and $h = 22$ ft.

$$\therefore \text{Required volume} = \frac{22}{3}[a^2 + b^2 + \sqrt{a^2 b^2}]$$

$$= \frac{22}{3}\left[\left(\frac{23}{3}\right)^2 + \left(\frac{9}{2}\right)^2 + \frac{23}{3} \times \frac{9}{2}\right]\text{ cub.ft}$$

$$= \frac{22}{3}\left[\frac{2116 + 729 + 1242}{36}\right]\text{ cub.ft}$$

Ex 4. [2009] How many litres of water will a bucket hold if its depth is 25 cm and radii on its top and bottom are 15 cm and 10 cm respectively.

Solution. A bucket resembles the frustum of a cone.

$\therefore$ Volume of the bucket $= \frac{\pi h}{3}[r_1^2 + r_2^2 + r_1 r_2]$;

Here, $r_1 = 15\text{cm}$, $r_2 = 10$ cm, $h = 25$ cm

$\therefore$ Required volume of water $= \frac{22}{7} \times \frac{25}{3}\left[15^2 + 10^2 + 15 \times 10\right]$ cu cm

$= \frac{550}{21} \times 475$ cu cm

$= 12440.48$ cu cm

$= \frac{12440.48}{1000}$ litres, $\quad$ 1 cu cm = 1 ml.

$= 12.440$ litres.

Ex 5. [2010] The circumference of the ends of a frustum of a cone are 44cm and 88 cm. If the height of the frustum be 14 cm, find its volume and curved surface area.

Solution: Let r_1, r_2 be the radius of the two ends, h be the height and s be the slant height.

Then, $2\pi r_1 = 88\text{cm}$, $2\pi r_2 = 44\text{cm}$, $h = 14\text{cm}$.

$\Rightarrow r_1 = \frac{88 \times 7}{2 \times 22} = 14\text{cm}$, $r_2 = \frac{44 \times 7}{2 \times 22} = 7\text{cm}$ and

$$s = \sqrt{h^2 + (r - r)^2} = \sqrt{14^2 + (14 - 7)^2} = 15.7\text{cm}$$

$\therefore$ Volume $= \frac{1}{3}\pi\{r_1^2 + r_2^2 + r_1 r_2\}h = \frac{1}{3} \times \frac{22}{7}[(14)^2 + (7)^2 + 14 \times 7] \times 14$ cub.cm

$= \frac{44}{3}[196 + 49 + 98]$ cub.cm $= 5030.67$ cub.cm

$\therefore$ Curved Surface Area $= \pi(r_1 + r_2)s = \frac{22}{7} \times (14+7) \times 15.7$ sq.cm $= 1036.2$sq.cm

Ex 6. [2010 A reservoir of of depth 10m is bounded by two rectangular faces of area 20×10sq.m and 30×15 sq. m. Find how much water in litres the reservoir can hold.

Solution: The reservoir is the frustum of a pyramid ;

Volume of the reservoir $= \frac{h}{3}\left[A_1 + A_2 + \sqrt{A_1 + A_2}\right]$

$A_1 = 30 \times 15$ sq. m $= 450$ sq. m ; $A_2 = 20 \times 10$ sq. m $= 200$ sq. m, $h = 10$m

Required volume $= \frac{10}{3}[450 + 200 + \sqrt{450 \times 200}]$

$= 3166.67$ cub.m , 1 cub. m $= 1000$ litre

$= 3166.67 \times 1000$ litre $= 3166670$ litre

Ex 7. The frustum of a square pyramid 20 ms high, 4m square at the base and 3 ms square at the top is capped at the top by a square pyramid 3 ms high. Find the volume of the solid so formed.

Solution: The solid is a combination of two parts, a frustum and a pyramid at the top. We first find the volume of each one of them separately, the sum of them is the required volume.

Volume of the pyramid at the top: $V_1 = \frac{1}{3}Ah$,

Here, $A = 3^2$ sq. m $= 9$ sq. m, $h = 3$m.

$\therefore V_1 = \frac{1}{3} \times 9 \times 3$cub. m $= 9$ cub. m.

Volume of the frustum: $V_2 = \frac{h_1}{3}[A_1 + A + \sqrt{A_1 A}]$

Now, $A = 9$ sq. m, $A_1 = 4^2$ sq. m $= 16$ sq. m, $h_1 = 20$ m.

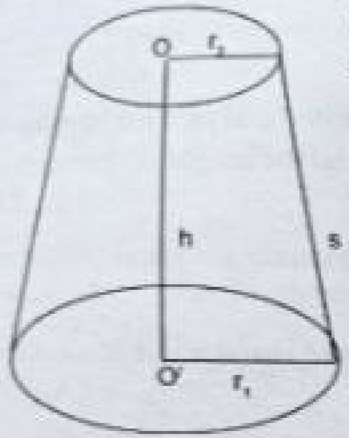
$\therefore V_2 = \frac{20}{3}\left[16 + 9 + \sqrt{16 \times 9}\right]$cub. m

$= \frac{20 \times 37}{3}$ cub. m $= 246.67$ cub. m

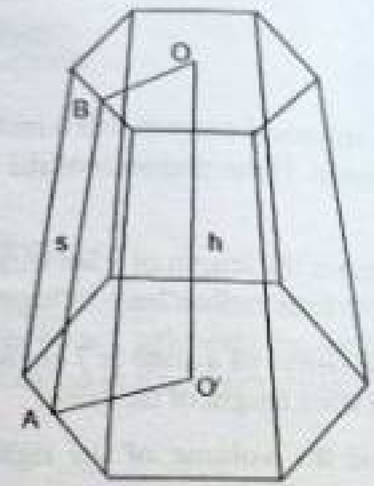
Therefore, required volume $= V_1 + V_2 = [9 + 246.67]$ cub. m $= 255.67$ cub. m

Ex 8. Find the quantity of water required to fill up a tank in the shape of a prismoid, the depth of which is 3 metres and the top and bottom are two parallel rectangles measuring 40×8 sq. m and 30×6 sq. m respectively.

h → height
s → slant height



Frustum of a Right Circular Cone



Frustum of a Right Pyramid

television tower, light house, bucket are some Examples of frustum of a cone.

Embankment, reservoir, brick chimney are some Examples of frustum of a pyramid.

As we have discussed earlier, cone is a special case of pyramid. *Comparing the two frustums, we observe that, in the frustum of the right pyramid, A and B being mid points, OA and OB are the radii of the inscribed circles of the base and the top respectively.*

Volume and Surface Area:

(a) For the Frustum of a Right Regular Pyramid:

(i) Volume = $\frac{h}{3}$ [$A_1+A_2+\sqrt{A_1A_2}$] cub. unit; A_1 = area of the base,
A_2 = area of the top, h = height.

(ii) Lateral Surface Area = $\frac{s}{2}$ [P + p] sq.unit; P = perimeter of the base,
p = perimeter of the top, s = slant height.

(iii) Total Surface Area = [$\frac{s}{2}$ (P + p) + A_1 + A_2]sq. unit

(b) For the Frustum of a Right Circular Cone:

(i) Volume = $\frac{\pi h}{3}$ [$r_1^2+r_2^2+r_1r_2$]cub.unit; r_1 = radius of the base,
r_2 = radius of the top, h = height.

(ii) Lateral Surface Area = $\frac{s}{2}$ [C + c]; C, c = circumference of the base and top respectively.
= $\pi s\,[r_1+r_2]$ sq. unit; s = slant height.

(iii) Total Surface Area = $\pi\,[\,s(r_1+r_2)+r_1^2+r_2^2]$ sq. unit;

(iv) Slant height of the frustum; $s=\sqrt{h^2+(r_1-r_2)^2}$

Worked out examples:

Ex 1. [2009, 2017] Find the volume of the frustum of a pyramid, the areas of whose ends are 32 sq. m and 20 sq. m and height is 6 m.

Solution: Volume of the frustum of a pyramid = $\frac{h}{3}$ [$A_1+A_2+\sqrt{A_1A_2}$]

Given, A_1 = 32 sq. m, A_2 = 20 sq. m, h = 6m.

Therefore, required volume = $\frac{6}{3}$ [$32+20+\sqrt{32\times 20}$]cub. m = 2[52 + 25.3]cub.m
= 154.60 cub.m

Ex 2. Find the area of the curved surface of the frustum of a right circular cone of thickness 10 cm, the radii of the upper and lower ends being 6 cm and 10 cm respectively.

Solution: Curved surface area of the frustum of a right circular cone,

$S=\pi s(r_1+r_2)$, given, r_1 = 10 cm, r_2 = 6 cm, h = 10 cm,

∴ slant height = $s=\sqrt{h^2+(r_1-r_2)^2}=\sqrt{10^2+(10-6)^2}$ cm = $\sqrt{116}$ cm = 10.77 cm

∴ Required area: $S=\frac{22}{7}\times 10.77(10+6)$ sq. cm = 541.58 sq. cm

Ex 3. If R, r and h be 16, 8 and 6 cm respectively of a frustum of a right circular cone, find s.

Solution: Let R, r h and s denote the radius of base, radius of top, height and slant height respectively of the frustum of a right circular cone.

Given, R = 16 cm, r = 8 cm, h = 6 cm, to find 's'.

∴ $s=\sqrt{r^2+h^2}=\sqrt{8^2+6^2}=\sqrt{100}$ = 10 cm

Ex 3. Prove that $(-\frac{1}{14}, \frac{39}{14})$ is the centre of the circle circumscribing the triangle whose angular points are (1, 1), (2, 3) and (−2, 2).

Solution: Let the points be $O(-\frac{1}{14}, \frac{39}{14})$, $A(1, 1)$, $B(2, 3)$ and $C(-2, 2)$.

We prove that $OA = OB = OC$.

$$OA^2 = (-\frac{1}{14}-1)^2 + (\frac{39}{14}-1)^2 = \frac{15^2+25^2}{14^2} = \frac{850}{14^2}$$

$$OB^2 = (-\frac{1}{14}-2)^2 + (\frac{39}{14}-3)^2 = \frac{29^2+3^2}{14^2} = \frac{850}{14^2}$$

$$OC^2 = (-\frac{1}{14}+2)^2 + (\frac{39}{14}-2)^2 = \frac{27^2+11^2}{14^2} = \frac{850}{14^2}$$

We have $OA = OB = OC$. Hence.

Ex 4. Prove that the points (−2, −11), is equidistant from (−3, 7) and (4, 6).

Solution: Let the points be $A(-2, -11)$, $B(-3, 7)$ and $C(4, 6)$.

$$AB^2 = (-2+3)^2 + (-11-7)^2 = 325$$
$$AC^2 = (-2-4)^2 + (-11-6)^2 = 325$$

Hence AB = AC.

Ex 5. If the ordinate of a point equidistant from (−3, 7) and (6, −11) be 9 then find its abscissa.

Solution: Let the points be A (x, 9).

It is equidistant from B(−3, 7) and C(6, −11).

So, $(x+3)^2 + (9-7)^2 = (x-6)^2 + (9+11)^2$

$\Rightarrow x^2 + 6x + 9 + 4 = x^2 - 12x + 36 + 400$

$\Rightarrow 18x = 423 \quad \Rightarrow x = \frac{47}{2}$

Ex 6. Prove that (−2, 3), (−3, 10) and (4, 11) are the angular points of an isosceles right angled triangle.

Solution: Let the points be A(−2, 3), B(−3, 10) and C(4, 11) are the given points.

$$AB^2 = (-2+3)^2 + (3-10)^2 = 50$$
$$AC^2 = (-2-4)^2 + (3-11)^2 = 100$$
$$BC^2 = (-3-4)^2 + (10-11)^2 = 50$$

We see AB = BC and $AB^2 + BC^2 = AC^2$ which implies ABC is an isosceles right angled triangle.

Ex 7. Find the coordinates of the point which divides the line joining the points (1, 3) and (2, 7) in the ratio 3: 4.

Solution: Let R(α, β) divides the line joining A(1, 3) and B(2, 7) in the ratio 3:4.

Then $\alpha = \frac{3\times2+4\times1}{3+4} = \frac{10}{7}$ $\qquad \beta = \frac{3\times7+4\times3}{3+4} = \frac{33}{7}$

The point is $(\frac{10}{7}, \frac{33}{7})$.

Ex 8. Find the coordinates of the point which divides internally and externally the line joining (−3, −4) to (−8, 7) in the ratio 7:5.

Solution: Let $R_1(\alpha_1, \beta_1)$ divides internally the line joining A(−3, −4) and B(−8, 7) in the ratio 7:5.

Then $\alpha_1 = \frac{7\times(-8)+5\times(-3)}{7+5} = -\frac{71}{12}$ $\quad \beta_1 = \frac{7\times7+5\times(-4)}{7+5} = \frac{29}{12}$

The point is $(-\frac{71}{12}, \frac{29}{12})$.

Let $R_2(\alpha_2, \beta_2)$ divides externally the line joining A(−3, −4) and B(−8, 7) in the ratio 7:5. Then

$\alpha_2 = \frac{7\times(-8)-5\times(-3)}{7-5} = -\frac{41}{2}$ $\quad \beta_2 = \frac{7\times7-5\times(-4)}{7-5} = \frac{69}{2}$

The point is $(-\frac{41}{2}, \frac{69}{2})$.

Ex 9. The line joining the points (1, −2) and (−3, 4) is trisected. Find the coordinates of the points of trisection.

Solution: A(1, −2) and B(−3, 4) are joined. AB is trisected by $R_1(\alpha_1, \beta_1)$ and $R_2(\alpha_2, \beta_2)$. So $AR_1 : R_1B :: 1:2$; $AR_2 : R_2B :: 2:1$

PQ is the diagonal of the triangle. So

$$PQ=\sqrt{PL^2+QL^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Thus we have distance between two points

(x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

And distance of a point (x_1, y_1) from the origin is $\sqrt{x_1^{\,2}+y_1^{\,2}}$

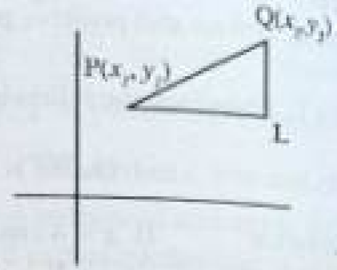


Fig. 1.4

4.1.5 Ratio Formula:

Point dividing the line joining two given points in a given ratio $m : n$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the plane. $R(x, y)$ is a point on PQ. R divides PQ in the ratio $m : n$.

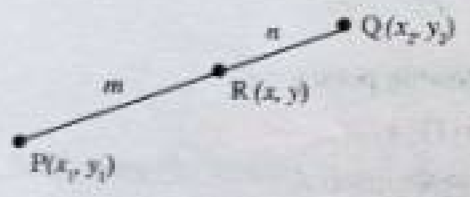


Fig 1.5

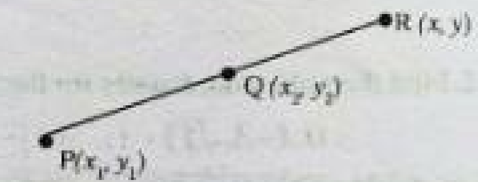


Fig. 1.6

Internal Division: R is between P and Q. In Fig. 1.5 $\dfrac{PR}{QR}=\dfrac{m}{n}$

Now $x=\dfrac{mx_2+nx_1}{m+n}$, $y=\dfrac{my_2+ny_1}{m+n}$

External Division : R is not in between P and Q. In Fig. 1.6 $\dfrac{PR}{QR}=\dfrac{m}{n}$

Now, $x=\dfrac{mx_2-nx_1}{m-n}$, $y=\dfrac{my_2-ny_1}{m-n}$

4.1.6 Area of a triangle whose vertices are given in Cartesian coordinates

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are vertices of a triangle ΔABC then area of the triangle is

$$\Delta ABC=\frac{-1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\} \qquad A$$

When we know about three points on a plane then it happens that either they are collinear or they form a triangle. If they are collinear, the simplification A will come out as zero. So to establish collinearity of three given points we may go through area formula A. if it is zero we may conclude that points are collinear.

Worked out Examples :

Ex 1. Prove that $(-1,-1)$, $(1, 1)$, $(-\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle.

Solution: Let the points be $A(-1,-1)$, $B(1, 1)$ and $C(-\sqrt{3}, \sqrt{3})$

Then $AB^2=(-1-1)^2+(-1-1)^2=8$

$BC^2=(1+\sqrt{3})^2+(1-\sqrt{3})^2=8$

$CA^2=(-\sqrt{3}+1)^2+(\sqrt{3}+1)^2=8$

So $AB = BC = CA$. hence triangles are equilateral.

Ex 2. Prove that $(-2,-1)$, $(1, 0)$, $(4, 3)$ and $(1, 2)$ are at the vertices of a parallelogram.

Solution: Let the points be $A(-2,-1)$, $B(1, 0)$, $C(4, 3)$ and $D(1, 2)$

Then $AB^2=(-2-1)^2+(-1-0)^2=10$

$BC^2=(1-4)^2+(0-3)^2=18$

$CD^2=(4-1)^2+(3-2)^2=10$

$DA^2=(1+2)^2+(2+1)^2=18$

$CA^2=(4+2)^2+(3+1)^2=52$

$DB^2=(1-1)^2+(2-0)^2=4$

We see $AB = CD$, $BC = DA$ means opposite sides are equal.

The diagonals CA and DB are unequal. So they are not rectangle. Hence a parallelogram.

4.1.2 Polar Coordinates:

Let us consider a ray $\overrightarrow{OX}$ and another ray $\overrightarrow{OX'}$. $\overrightarrow{OX'}$ Initially coincides with $\overrightarrow{OX}$, then rotate about O. The measure of angle $X'OX$ is positive if rotation is anticlockwise and negative if rotation is clockwise. $\overrightarrow{OX}$ is called initial line and O is called Pole. If P is a point on the plane, the length $OP(r)$ is measured and angle POX (θ) is measured. The coordinate of P is (r, θ). r is called radius vector and θ is the vectorial angle. Fig. 1.2 explains.

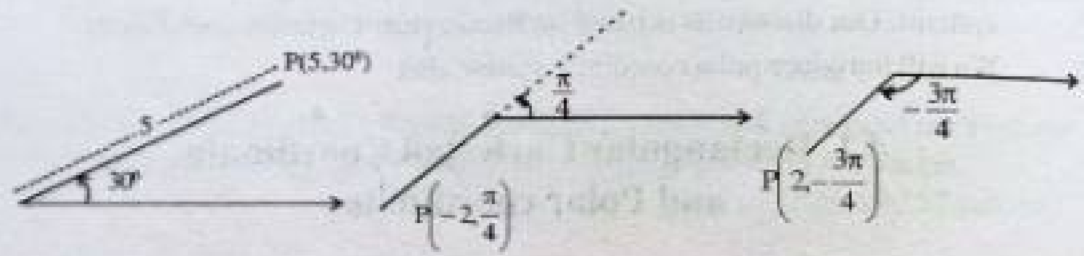


Fig1.2

4.1.3 Transformation of Cartesian Coordinates into polar coordinates and vice-versa:

If cartesian coordinates of P are (x, y) and Polar coordinates of P are (r, θ) then from the diagram it is clear that

$$x = r\cos\theta,\ y = r\sin\theta \quad \text{....... (1)}$$

or, $$r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\frac{y}{x} \quad \text{...... (2)}$$

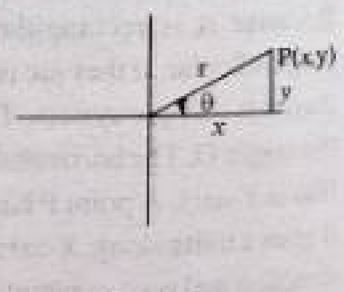


Fig. 1.3

When r and θ are given x and y can be found from (1) and when x and y are given r and θ can be found

from (2). Here we assume that the origin O of Cartesian system coincides with pole of polar system and positive part of X-axis acts as initial line of polar system.

Ex 1. Transform the following polar coordinates into rectangular Cartesian coordinates.

i) $(4, 30°)$ ii) $(2, -60°)$ iii) $\left(-1, \frac{\pi}{4}\right)$

Solution: i) $x = 4\cos 30° = 2\sqrt{3}$, $y = 4\sin 30° = 2$

Cartesian coordinates $(2\sqrt{3}, 2)$

ii) $x = 2\cos(-60°) = 1$, $y = 2\sin(-60°) = -\sqrt{3}$

Cartesian coordinates $(1, -\sqrt{3})$

iii) $x = -1\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, $y = -1\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Cartesian coordinates $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Ex 2. Find the polar coordinates for the following points:

i) $(-3, \sqrt{3})$ ii) $(-1, -1)$

Solution: i) $r = \sqrt{(-3)^2 + 3} = \sqrt{12} = 2\sqrt{3}$

$\theta = \tan^{-1}\frac{\sqrt{3}}{-3} = \tan^{-1} - \frac{1}{\sqrt{3}} = 150°$ since the point is on 2nd quadrant.

Polar coordinates $(2\sqrt{3}, 150°)$.

ii) $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\frac{-1}{-1} = 180° + 45° = 225°$ since the point is on 3rd quadrant.

Polar coordinates $(\sqrt{2}, 225°)$.

4.1.4 Distance Formula: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. From Fig. 1.4 we see that PQL is a right angled triangle right angled at L. $PL = x_2 - x_1$, $QL = y_2 - y_1$

(Hint. same as Ex. 3. here, $h = 7$ ft 6 inch $= \frac{15}{2}$ ft, $h_1 = 61$ft)

2. A television tower is in the form of a frustum of a right circular cone of height 35 metres. The ends have diameters 5m and 2m respectively. Find the volume of the tower. (Ans.357.5 cub.m)

3. The ends of the frustum of a pyramid are regular hexagons with sides 10 cm and 5 cm respt. and the slant height is 15 cm. Find the volume. (Ans.2176.54 cub. cm)

4. The slant height of the frustum of a right circular cone is 12 m and the radii of its ends are 15 m and 17 m. Find the volume of the frustum. (Ans. 9532.21 cub. m)

5. The height of a bucket is 8 cm and radii of its lower and upper end are 3 cm and 9 cm respectively. Find the volume of water required to fill the bucket. (Ans. 980.57 cub. cm)

* * *

4 COORDINATE GEOMETRY

Sometimes it becomes necessary to study motion of a particle in a plane. For this, we fix some coordinate systems and then with respect to that system we find location of points. There are different types of coordinate systems. Our discussion is based on Rectangular Cartesian coordinates. We will introduce polar coordinate system also.

4.1. Rectangular Cartesian Coordinate and Polar coordinate

4.1.1 Cartesian Coordinates:

The word Cartesian comes after the French Mathematician Descartes (1596-1640) who introduced this system. Here we consider two mutually perpendicular straight lines, called axes. Because it is rectangular, so they are mutually perpendicular. If they are not mutually perpendicular then it is oblique system. The point of intersection is the origin O. The horizontal line is X-axis and vertical line is Y-axis. A point P has coordinates (x, y) means it goes x units along X-axis from origin then changes direction and goes y units along Y-axis. First coordinate x is called abscissa and the second coordinate y is called ordinate.

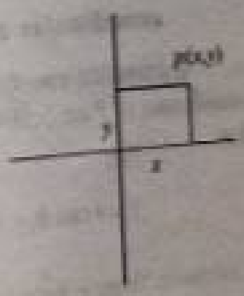


Fig 1.1

4.2.1 Standard forms of the equations of a straight line

A. Straight line parallel to the axes. BL is a straight line parallel to X-axis. For every point on BL its y coordinate is b. Equation of BL is $y = b$. AM is a straight line parallel to Y-axis. For every point on AM its x coordinate is a. Equation of AM is $x = a$.

When $b = 0$ then BL coincides with X-axis. So **equation of X-axis is $y = 0$**. When $a = 0$ then AM coincides with Y-axis. So **equation of Y-axis is $x = 0$.**

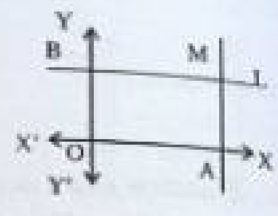


Fig. 2.4

B. Gradient form: Let L be straight line. It is given that it makes angle θ with positive direction of X-axis. The tangent of the angle is termed as m and called Gradient of the line L. L intersects with Y-axis at $(0, c)$. c is the y-intercept. We take $P(x, y)$ any point on L. From the diagram we see $y = CP$, $x = AB$, $\tan\angle PAB = m$. $xm = BP$, $BC = c$. So, $y = CP = BP + BC = mx + c$.

So, equation of the straight line having gradient m and y-intercept c is given by $y = mx + c$

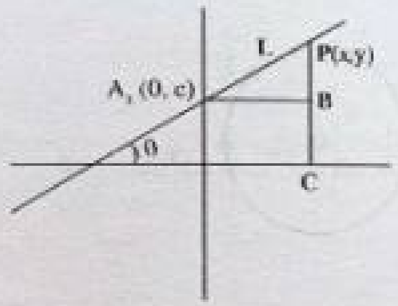


Fig 2.5

Corollary 1. Let (x_1, y_1) be a point on the line L with equation $y = mx + c$. Then (x_1, y_1) satisfies the equation $y = mx+c$ and we have $y = mx+c$, $y_1 = mx_1 + c \Rightarrow y - y_1 = m(x - x_1)$

So, equation of the straight line having gradient m and passing through (x_1, y_1) is $y - y_1 = m(x - x_1)$

Corollary 2. Let (x_1, y_1), (x_2, y_2) be two points on the line L with equation $y = mx + c$. Then (x_1, y_1) and (x_2, y_2) satisfy the equation $y = mx + c$ and we have

$$y = mx + c,\quad y_1 = m x_1 + c,\ y_2 = m x_2 + c$$

$$\Rightarrow y - y_1 = m(x - x_1)\ ,\quad y - y_2 = m(x - x_2),$$

hence $y_2 - y_1 = m(x_2 - x_1)$

From above we get $m = \dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Without knowing m the relation can be stated.

So, equation of the straight line passing through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or, $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

C. Intercept form: L is a straight line. L intersects X-axis and Y-axis at A and B respectively. $OA = a$ and $OB = b$. So a is x-intercept or intercept on X-axis and b is y-intercept or intercept on Y-axis. A is $(a, 0)$ and B is $(0, b)$. Let $P(x, y)$ be current point on L. So, by above result equation to the line passing through A and B is given by

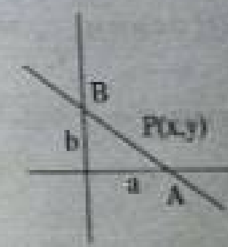


Fig. 2.6

$$y - 0 = \frac{b - 0}{0 - a}(x - a)$$

$$\Rightarrow ya + bx = ab$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

Thus the intercept form of straight line L having x-intercept a and y-intercept b is given by $\dfrac{x}{a} + \dfrac{y}{b} = 1$

D. Perpendicular form: The length of perpendicular to line L from the origin O is p. This perpendicular makes angle α with X-axis. From diagram it is clear that intercept of L on X-axis is $p \sec \alpha$ and on Y-axis is p cosec α. Accordingly equation of L is given by

15. Change into cartesian coordinates-

i) $r^2 \sin 2\theta = 2a^2$ ii) $r = 4$ iii) $\theta = 45°$

iv) $r^2 = \cos 2\theta$ v) $\sqrt{r}\cos\frac{\theta}{2} = \sqrt{a}$

Answer

1. i) 3 ii) $\sqrt{29}$ iii) 1 iv) $\sqrt{53}$ v) $\sqrt{90}$

2. i) $(\frac{20}{7}, \frac{15}{7})$ ii) $(-4, 9)$

3. i) $(\frac{11}{3}, \frac{17}{3})$ ii) $(13, 15)$

6. i) 1 ii) 29 iii) 11 iv) 9 v) $\frac{5}{2}$

7. $\frac{11}{8}$ 8. $\sqrt{34}$, $\sqrt{26}$

14. i) $\theta = \alpha$ ii) $r^2 = a^2 \cos 2\theta$ iii) $2 \tan 2\theta = 1$

iv) $r \sin^2\theta = 8\cos\theta$ v) $\sin 2\theta = 1$

15. i) $xy = a^2$ ii) $x^2 + y^2 = 16$

iii) $y = x$ iv) $(x^2 + y^2)^2 = x^2 - y^2$ v) $y^2 = 4a(a - x)$

4.2 Straight Line

Equation and Locus: If an equation in x and y be given, then the Cartesian coordinates of a number of points satisfy the equation. These points constitute a path. This path is defined as locus represented by the given equation. In other words, when the point moves in such a way that it satisfies a given condition, the path it traces out is called its locus.

Fig.2.1

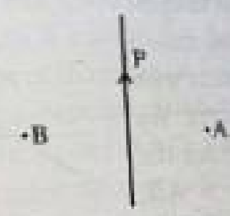


Fig.2.2

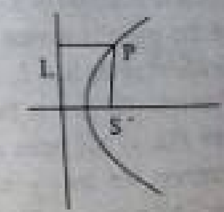


Fig.2.3

Example 1. O is a fixed point. P is the moving point. P moves in such a way that its distance from O is a constant. This locus is a circle with centre O (Fig.2.1)

Example 2. A, B are two fixed points. P moves in such a way that its distances from A and B are equal. The locus is a straight line (Fig. 2.2).

Example 3. S is a fixed point. L is a fixed line. P moves in such a way that its distance from S is equal to its distance from L. This locus is a parabola (Fig. 2.3).

The locus of a point satisfying the first-degree equation in x and y is a straight line. This means the first-degree equation $Ax + By + C = 0$ represents a straight line.

ii) $r^2\cos 2\theta = a^2 \Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = a^2 \Rightarrow x^2 - y^2 = a^2$

iii) $\theta = \tan^{-1} m \Rightarrow \tan\theta = m \Rightarrow \dfrac{y}{x} = m \quad \Rightarrow y = mx$

Ex 15. Let A $(2, 60^\circ)$ and B$(1, 30^\circ)$ be two points. Find line segments AB.

Solution: Let O be the origin. OAB is a triangle such that OA = 2, OB = 1 and

$\angle AOB = \angle AOX - \angle BOX = 30^\circ$.

So, $AB^2 = OA^2 + OB^2 - 2.OA.OB\cos\angle AOB = 2^2 + 1^2 - 2.2.1.\cos 30^\circ$.

$\Rightarrow AB = \sqrt{5-2\sqrt{3}}$

Alternately A $(2, 60^\circ)$ and B$(1, 30^\circ)$ have Cartesian coordinates

A$(1, \sqrt{3})$ and B$(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2})$

$$AB^2 = (1-\frac{\sqrt{3}}{2})^2 + (\sqrt{3}-\frac{1}{2})^2 = \frac{1}{4}\{(7-4\sqrt{3}) + (13-4\sqrt{3})\} = 5-2\sqrt{3}$$

$\Rightarrow AB = \sqrt{5-2\sqrt{3}}$

Ex 16. [2013] A$(a\sec\theta, b\tan\theta)$, B$(a\tan\theta, -b\sec\theta)$ are any two points and O is the origin, show that $OA^2 - OB^2 = a^2 - b^2$

Solution: a) $OA^2 = (a\sec\theta)^2 + (b\tan\theta)^2 = a^2\sec^2\theta + b^2\tan^2\theta$

$OB^2 = (a\tan\theta)^2 + (-b\sec\theta)^2 = a^2\tan^2\theta + b^2\sec^2\theta$

Hence $OA^2 - OB^2 = a^2\sec^2\theta + b^2\tan^2\theta - (a^2\tan^2\theta + b^2\sec^2\theta) = a^2 - b^2$

Exercise 4.1

1. Find the distance between the following points:

i) $(-2, 0)$, $(1, 0)$ ii) $(4, 6)$, $(2, 1)$ iii) $(\sqrt{a}, 1)$, $(\sqrt{a}, 2)$

iv) $(1, 5)$, $(3, -2)$ v) $(6, 5)$, $(-3, 8)$

2. Divide the line joining $(2, 3)$ and $(4, 1)$ in the ratio 3 : 4 i) internally ii) externally
3. Divide the line joining $(-1, 1)$ and $(6, 8)$ in the ratio 2 : 1
i) internally ii) externally
4. Prove that $(1, 2)$, $(-4, 3)$, $(11, 0)$ are collinear.
5. Prove that $(4, 2)$, $(7, 5)$, $(9, 7)$ lie on a straight line.
6. Find the area of the triangle whose vertices are
i) $(0, 4)$, $(3, 6)$, $(-8, -2)$
ii) $(5, 2)$, $(-9, -3)$, $(-3, -5)$
iii) $(1, 4)$, $(3, -2)$, $(-3, 5)$
iv) $(1, 1)$, $(3, 4)$, $(5, -2)$
v) $(0, 1)$, $(1, -1)$, $(2, 2)$
7. Coordinates of A, B, C, D are $(6, 3)$, $(-3, 5)$, $(4, -2)$ and $(x, 3x)$ respectively. If the area of ΔABC be twice the area ΔDBC, of then find the value of x.
8. Show that $(0, 0)$, $(4, 3)$, $(3, 5)$, $(-1, 2)$ form a parallelogram and find the lengths of its diagonals.
9. Show that the points $(2, 1)$, $(-2, 4)$, $(-5, 0)$, $(-1, -3)$ are the vertices of a square.
10. Show that the points $(a, bc-a^2)$, $(b, ca-b^2)$, $(c, ab-c^2)$ are collinear.
11. The vertices of two triangles are $(3, 0)$, $(0, 7)$, $(1, 1)$ and $(13, 3)$, $(2, 3)$, $(-11, 2)$ respectively. Show that they have same area and same centroid.
12. ABCD is a square and P is any point on the plane. Prove that
$$PA^2 + PC^2 = PB^2 + PD^2$$
(take vertices of the square as $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$ and $P(x, y)$)
13. Lay down the points:
i) $(3, 60^\circ)$, ii) $(-1, 50^\circ)$ iii) $(-2, 230^\circ)$ iv) $(4, -160^\circ)$ v) $(2, -\frac{\pi}{4})$
14. Change into polar coordinates-
i) $y = x\tan\alpha$ ii) $(x^2+y^2)^2 = a^2(x^2-y^2)$ iii) $x^2 = 4xy + y^2$
iv) $y^2 = 8x$ v) $x^2 + y^2 = 2xy$

$$\alpha_1=\frac{1\times(-3)+2\times1}{1+2}=-\frac{1}{3} \qquad \beta_1=\frac{1\times4+2\times(-2)}{1+2}=0$$

Thus, R_1 is $(-\frac{1}{3},0)$.

$$\alpha_2=\frac{2\times(-3)+1\times1}{2+1}=-\frac{5}{3} \qquad \beta_2=\frac{2\times4+1\times(-2)}{2+1}=2$$

R_2 is $(-\frac{5}{3},2)$.

Ex 10. If the vertices of a triangle are (x_1,y_1), (x_2,y_2) and (x_3,y_3), then show that the coordinates of the centroid are $(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3})$

Solution: Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of the triangle ABC. Let M be the mid point of BC and G be its centroid. Then coordinates of M are $(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2})$. As G is the centroid, G divides AM in the ratio 2:1. So, $G(\alpha,\beta)$

is given by $\alpha=\dfrac{2\times\frac{1}{2}(x_2+x_3)+1\times x_1}{2+1}=\dfrac{x_1+x_2+x_3}{3}$

$$\beta=\frac{2\times\frac{1}{2}(y_2+y_3)+1\times y_1}{2+1}=\frac{y_1+y_2+y_3}{3}$$

In a similar way it can be shown that points of trisection for the other medians coincide with $(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3})$. Hence the result.

Ex 11. If the coordinates of the vertices of a triangle ΔABC be $A(3,0)$, $B(0,6)$ and $C(6,9)$ and if D and E divide AB and AC respectively internally in the ratio 1:2 then prove that the area of triangle is $\Delta ABC=9\times\Delta ADE$.

Solution: D divides AB in the ratio 1: 2. So D is given by

$$(\frac{2\times3+1\times0}{1+2},\frac{2\times0+1\times6}{1+2})\Rightarrow D(2,2)$$

E divides AC in the ratio 1: 2. So E is given by

$$(\frac{2\times3+1\times6}{1+2},\frac{2\times0+1\times9}{1+2})\Rightarrow E(4,3)$$

$$\Delta ABC=\frac{1}{2}\{3(6-9)+0(9-0)+6(0-6)\}=-\frac{45}{2}$$

$$\text{Again, } \Delta ADE=\frac{1}{2}\{3(2-3)+2(3-0)+4(0-2)\}=-\frac{5}{2}$$

Though we get negative values, we reject the negative sign, since only magnitude is required for area. It is now clear that $\Delta ABC=9\times\Delta ADE$. Proved.

Ex 12. If two vertices of a triangle are (2, 7) and (6, 1) and its centroid is (6, 4), then find the third vertex.

Solution: Let (a,b) be the third vertex. According to definition

$$6=\frac{2+6+a}{3}\Rightarrow a=10$$

$$4=\frac{7+1+b}{3}\Rightarrow b=4$$

The third vertex is (10, 4).

Ex 13. Change into polar coordinates-

i) $x^2+y^2=k^2$ ii) $x^2-y^2=6x$ iii) $xy=9$

Solution: The relation is $x=r\cos\theta$, $y=r\sin\theta$

i) $x^2+y^2=k^2\Rightarrow(r\cos\theta)^2+(r\sin\theta)^2=k^2\Rightarrow r^2=k^2\Rightarrow r=k$

ii) $x^2-y^2=6x\Rightarrow(r\cos\theta)^2-(r\sin\theta)^2=6r\cos\theta\Rightarrow r\cos2\theta=6\cos\theta$

iii) $xy=9\Rightarrow r\cos\theta\, r\sin\theta=9\Rightarrow r^2\sin2\theta=18$

Ex 14. Change into cartesian coordinates-

i) **[2012, 2016]** $r=a\sin2\theta$ ii) $r^2\cos2\theta=a^2$ iii) $\theta=\tan^{-1}m$

Solution: The relation is $r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$

i) $r=a\sin2\theta=2a\sin\theta\cos\theta\Rightarrow r^2=4a^2\sin^2\theta\cos^2\theta$

$\Rightarrow r^6=(r^2)^3=4a^2r^2\sin^2\theta\, r^2\cos^2\theta\Rightarrow(x^2+y^2)^3=4a^2x^2y^2$

$$\Rightarrow (x-a)^2+y^2+(x+a)^2+a^2-2\sqrt{(x-a)^2+y^2}\sqrt{(x+a)^2+y^2}=c^2$$
$$\Rightarrow 2(x^2+y^2+a^2)-c^2=-2\sqrt{(x^2-a^2)^2+2y^2(x^2+a^2)+y^4}$$
$$\Rightarrow \{2(x^2+y^2+a^2)-c^2\}^2=4\{(x^2-a^2)^2+2y^2(x^2+a^2)+y^4\}$$
$$\Rightarrow \{2(x^2+y^2+a^2)-c^2\}^2=4\{(x^2+a^2)^2+2y^2(x^2+a^2)+y^4-4x^2a^2\}$$
$$\Rightarrow 4(x^2+y^2+a^2)^2-4(x^2+y^2+a^2)c^2+c^4=4\{(x^2+a^2+y^2)^2-4x^2a^2\}$$
$$\Rightarrow 4x^2(c^2-4a^2)+4c^2y^2=c^2(c^2-4a^2)$$
$$\Rightarrow \frac{x^2}{\frac{c^2}{4}}+\frac{y^2}{\frac{c^2-4a^2}{4}}=1$$

Ex 5. If A, B, C are three points whose coordinates are $(a, 0)$, $(-a, 0)$ and $(c, 0)$ respectively; if P be a movable point such that $PA^2+PB^2=2PC^2$ find locus of P.

Solution: $PA^2=(x-a)^2+y^2$

$PB^2=(x+a)^2+y^2$

$PC^2=(x-c)^2+y^2$

By given condition $(x-a)^2+y^2+(x+a)^2+y^2=2\{(x-c)^2+y^2\}$

$$\Rightarrow 2x^2+2a^2=2x^2+2c^2-4cx$$
$$\Rightarrow x=\frac{c^2-a^2}{2c}$$

Ex 6. Find equation to the line which makes angle

i) 30° with X-axis and has y-intercept -4.

ii) 135° with X-axis and has y-intercept 2.

Solution: i) $m=\tan 30^\circ=\frac{1}{\sqrt{3}}$, $c=-4$. So equation is $y=\frac{1}{\sqrt{3}}x-4$

ii) $m=\tan 135^\circ=-1$, $c=2$.

So equation is $y=-x+2$. Equivalently $y+x-2=0$

Ex 7. Find equation to the line having gradient

i) 2 and passing through $(0, 4)$.

ii) 1 and passing through $(-2, 3)$.

Solution: i) $m=2 \Rightarrow m=2=\frac{y-4}{x-0}$. So equation is $y-2x-4=0$

ii) $m=1 \Rightarrow m=1=\frac{y-3}{x+2}$. So equation is $y-x-5=0$

Ex 8. Find equation to the line passing through

i) $(2, 1)$ and $(4, 6)$ ii) $(0, 0)$ and $(1, 2)$ iii) $(-3, 1)$ and $(3, 3)$.

Solution: i) $\frac{y-6}{x-4}=\frac{1-6}{2-4} \Rightarrow 2(y-6)=5(x-4) \Rightarrow 5x-2y-8=0$

ii) $\frac{y-0}{x-0}=\frac{2-0}{1-0} \Rightarrow 2x=y$ (It is to be noted that if origin is a point on the line then there is no constant term in the equation.)

iii) $\frac{y-3}{x-3}=\frac{1-3}{-3-3} \Rightarrow 6(y-3)=2(x-3) \Rightarrow x-3y+6=0$

Ex 9. The equation of a straight line is $3x+4y-1=0$. Write the

i) gradient form,

ii) intercept form and iii) perpendicular form.

Find length of perpendicular to the line from origin, $(2, 3)$ and $(-4, 1)$.

i) Gradient form is $y=-\frac{3}{4}x+\frac{1}{4}$

Gradient is $-\frac{3}{4}$ and y-intercept is $\frac{1}{4}$

ii) Intercept form is $\frac{x}{\frac{1}{3}}+\frac{y}{\frac{1}{4}}=1$

i) the gradient of the line joining P and P' is $\frac{B}{A}$ since it is perpendicular to $Ax+By+C=0$.

So $\frac{b-b'}{a-a'}=\frac{B}{A}$

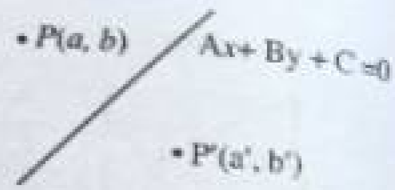


Fig. 2.10

ii) the distance of P from $Ax+By+C=0$ and distance of P' from $Ax+By+C=0$ are equal in magnitude but of opposite sign. i.e.

$$\frac{Aa+Bb+C}{\sqrt{A^2+B^2}}=-(-\frac{Aa'+Bb'+C}{\sqrt{A^2+B^2}})$$

Since A, B, C, a, b are known, from above two relations $P'(a', b')$ can be found. Some simple examples are reflection of $(2, 3)$ on Y-axis is $(-2, 3)$ and its reflection on X-axis is $(2, -3)$.

Worked out Examples

Ex 1. Find locus of the point moving at an equal distance from $(2, 4)$ and $(3, 1)$.

Solution:
$$(x-2)^2+(y-4)^2=(x-3)^2+(y-1)^2$$
$$\Rightarrow x^2+y^2-4x-8y+20=x^2+y^2-6x-2y+10$$
$$\Rightarrow 2x-6y+10=0$$
$$\Rightarrow x-3y+5=0$$

Ex 2. Find locus of the point moving at an equal distance from $(1, 0)$ and $(0, -2)$.

Solution:
$$(x-1)^2+(y-0)^2=(x-0)^2+(y+2)^2$$
$$\Rightarrow x^2+y^2-2x+1=x^2+y^2+4y+4$$
$$\Rightarrow 2x+4y+3=0$$

Ex 3. Find locus of the point marked on a rod of length $a+b$ and dividing it into sections of length a and b respectively; the rod is sliding on the coordinate axes in the positive quadrant.

Solution: We observe the diagram.

$P(x, y)$ is the moving point. The rod is of length $a+b$ and it meets X-axis at $A(\alpha, 0)$, Y-axis at $B(0, \beta)$. So $P(x, y)$ divides the line segment joining $A(\alpha, 0)$ and $B(0, \beta)$ in the ratio $b:a$.

$$\Rightarrow x=\frac{a\alpha+b.0}{a+b}=\frac{a\alpha}{a+b},$$
$$y=\frac{a.0+b\beta}{a+b}=\frac{b\beta}{a+b}$$
$$\Rightarrow \alpha=\frac{a+b}{a}x\ ,\ \beta=\frac{a+b}{b}y$$

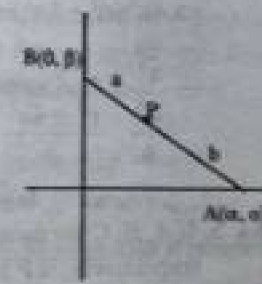


Fig. 2.11

On the other hand ΔAOB is a right angled triangle.

So $\alpha^2+\beta^2=(a+b)^2$

$$\Rightarrow (\frac{a+b}{a}x)^2+(\frac{a+b}{b}y)^2=(a+b)^2$$
$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Thus we get the moving point $P(x, y)$ follows the path traced by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Hence the locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Ex 4. $A(a, 0)$ and $B(-a, 0)$ are two points. P moves so that $PA+PB=c$. Find locus of P.

Solution: Let $P(x, y)$ is the moving point.

$$PA+PB=c$$
$$\Rightarrow \sqrt{(x-a)^2+y^2}+\sqrt{(x+a)^2+y^2}=c$$

4.2.3 Collinearity

Condition that the three given points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear. Collinearity can be tested in the following ways:

i) We look for the area of the triangle with the given points as vertices. As they are collinear so the triangle does not exist; means area we get zero.

ii) We find length of three different line segments with these points as end points. If there are two segments such that their sum equals the third one then the points are collinear.

iii) The gradient of the two lines one passing through (x_1, y_1) and (x_2, y_2) and the other through (x_2, y_2) and (x_3, y_3) must be equal.

Also other ways are there.

4.2.4 Points of intersection of two straight lines

If $Ax + By + C = 0$ and $ax + by + c = 0$ are two straight lines then either they are parallel or they will intersect at some point. If points of intersection is (α, β), then α, β satisfy both the equations. Thus α, β are the solutions of the two simultaneous equations

$$Ax + By + C = 0 \text{ and } ax + by + c = 0.$$

4.2.5 Equation of a straight line through the point of intersection of two given lines

Whenever there are two straight lines, there exists a point of intersection provided coefficients are not proportional, i.e. $Ax + By + C = 0$ and $ax + by + c = 0$ have point of intersection (x_1, y_1) if $\frac{a}{A} \neq \frac{b}{B}$. The equation to the straight line passing through (x_1, y_1) is $\frac{y - y_1}{x - x_1} = m$. m is any constant which determines the slope.

Also in another way we can write the equation as

$$Ax + By + C + k\,(ax + by + c) = 0$$

Note: Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

4.2.6 Angle between two given lines :

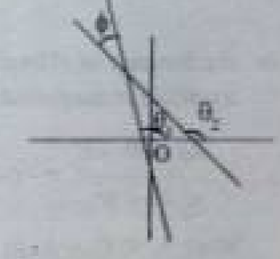


Fig. 2.9

If m_1 and m_2 are gradients of the two lines then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$; θ_1 and θ_2 are the angles made by the lines with X-axis.

Angle between them is now $\phi = \theta_2 - \theta_1$

$$\tan(\theta_2 - \theta_1) = \frac{m_2 - m_1}{1 + m_1 m_2}$$

4.2.7 Condition of Perpendicularity and Parallelism:

If $\theta_2 = \theta_1$ then two lines are parallel. Then $m_1 = m_2$

If $\theta_2 - \theta_1 = \frac{\pi}{2}$ then two lines are perpendicular.

Then $m_1 . m_2 = -1$.

We can say a line parallel to $Ax + By + C = 0$ is $Ax + By + k = 0$ (coefficients are equal)

And a line perpendicular to $Ax + By + C = 0$ is $Bx - Ay + k = 0$

4.2.8 Reflection of a point on a line:

Let $Ax + By + C = 0$ be a line and $P(a, b)$ be a point.

P is reflected on the line. The reflected point is $P'(a', b')$. Two things happen–

$$\frac{x}{p\sec\alpha}+\frac{y}{p\,\mathrm{cosec}\,\alpha}=1$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p.$$

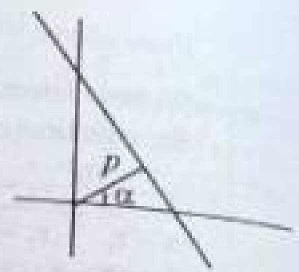


Fig. 2.7

Now we have learnt that the first-degree equation in x and y is of the form $Ax + By + C = 0$ represents a straight line.

Its gradient form is $y=-\frac{A}{B}x-\frac{C}{B}$. Therefore,

gradient of the line is $-\frac{A}{B}$ and intercept on Y-axis is $-\frac{C}{B}$.

Intercept form of straight line is $\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1$

Its intercept on X-axis is $-\frac{C}{A}$ and on Y-axis is $-\frac{C}{B}$.

Perpendicular form of straight line is $-\frac{Ax}{\sqrt{A^2+B^2}}-\frac{By}{\sqrt{A^2+B^2}}=\frac{C}{\sqrt{A^2+B^2}}$

This is because $\sin^2\alpha+\cos^2\alpha=1$ which is fulfilled only when coefficient of x is $\frac{A}{\sqrt{A^2+B^2}}$ and coefficient of y is $\frac{B}{\sqrt{A^2+B^2}}$. It is to be noted that to make p positive sign of the square root is accordingly chosen.

4.2.2 Distance of a line from a point

Length of perpendicular from a given point (x', y') to a given straight line $Ax + By + C = 0$;

Let us see the diagram. L is the line $Ax+By+C=0$. $P(x', y')$ be the point. L' is another line through $P(x', y')$ parallel to L. PN is distance of P from L. OM is perpendicular from O to L. OM is extended to Q on L'. $NP = MQ$.

$$OM = p = \frac{C}{\sqrt{A^2+B^2}}$$

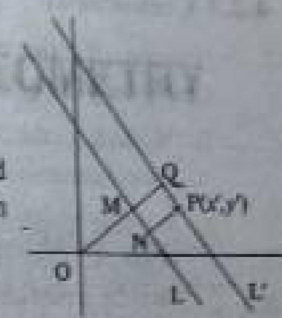


Fig. 2.8

Since OQ makes same angle as OM with X-axis and OQ is perpendicular distance of L' from origin, so equation of L' is $-\frac{Ax}{\sqrt{A^2+B^2}}-\frac{By}{\sqrt{A^2+B^2}}=OQ$

Since (x', y') is a point on L'

so $-\frac{Ax'}{\sqrt{A^2+B^2}}-\frac{By'}{\sqrt{A^2+B^2}}=OQ$

PN = distance of P from $L = OQ - OM$

$$=-\frac{Ax'}{\sqrt{A^2+B^2}}-\frac{By'}{\sqrt{A^2+B^2}}-\frac{C}{\sqrt{A^2+B^2}}$$

$$=-\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}$$. We consider the magnitude only.

It is important to note that the perpendicular from origin on a line is always taken positive. So if C of $Ax+By+C=0$ is negative then for $\sqrt{A^2+B^2}$ negative sign should be chosen. For a point (α, β) the perpendicular from (α, β) to $Ax+By+C=0$ is positive if it is on the same side as $(0, 0)$.

Consider the line $x-y+1=0$. $(1, 1)$ lies on the same side as origin and $(1, 3)$ lies on the opposite side. Distance of $(1, 1)$ to $x-y+1=0$ is $\frac{1-1+1}{\sqrt{1^2+1^2}}=\frac{1}{\sqrt{2}}$

Distance of $(1, 3)$ to $x-y+1=0$ is $\frac{1-3+1}{\sqrt{1^2+1^2}}=-\frac{1}{\sqrt{2}}$

Miscellaneous Worked out Examples:

Ex 1. [2009] i) What is the slope of the line $y = x$?

ii) Find the length of perpendicular from the point (3, 2) to the straight line $3x - 4y + 9 = 0$.

iii) In what ratio the line joining the points (3, 4) and (5, – 7) will be divided by X-axis?

iv) **[2010]** Find the distance between the pair of parallel lines $4x + 3y - 5 = 0$ and $4x + 3y + 7 = 0$.

v) A straight line is at a distance $\frac{1}{2}$ from the origin and it passes through the point (0, 1). Find its equation.

vi) Prove that the line passing through the points (8, – 4) and (– 5, 6) is perpendicular to the straight line $13x - 10y + 20 = 0$.

Solution: i) 1 ii) $\frac{3\times3-4\times2+9}{\sqrt{3^2+4^2}} = 2$

iii) At that point y coordinate is 0. Let the ratio be $m : n$.

$0 = \frac{4n + (-7).m}{m+n} \Rightarrow m : n$ is 4 : 7.

iv) Let us choose one point from $4x+3y-5=0.(2,-1)$ is a point on $4x+3y-5=0$. Distance of the point from $4x + 3y + 7 = 0$ is same as the distance between the two lines. It is $\frac{4\times2+3\times(-1)+7}{\sqrt{4^2+3^2}} = \frac{12}{5} = 2.2$

v) The perpendicular form is $x \cos\alpha + y \sin\alpha = p$. $p = \frac{1}{2}$. It passes through (0, 1)

$\Rightarrow 0 + 1.\sin\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$ or, $\frac{5\pi}{6} \Rightarrow \cos\alpha = \frac{\sqrt{3}}{2}$, or $-\frac{\sqrt{3}}{2}$.

The equation is $\sqrt{3}x + y = 1$ or, $-\sqrt{3}\,x + y = 1$.

vi) Equation to the line passing through (8, – 4) and (– 5, 6) is

$\frac{y+4}{x-8} = \frac{6+4}{-5-8} = -\frac{10}{13}$

$\Rightarrow 10x + 13y = 28$. So it is perpendicular to $13x - 10y + 20 = 0$.

Ex 2. [2009] i) In what ratio does the origin divide the line segment joining the points (– 2, 0) and (4, 0)?

ii) If the points (3, 4) and $(a, 3)$ are equidistant from (– 1, 7) find the value of a.

iii) Show that the points (– 2, 3), (– 6, 1) and (– 10, – 1) are collinear.

iv) The polar equation of a curve is given by $r = 2a\cos\theta$. Find its Cartesian equation.

v) Find the equation of a straight line passing through (2, 3) and parallel to the straight line $2x - 3y + 8 = 0$

Solution:

i) At the origin coordinate is (0, 0). Let the ratio be $m : n$. For x coordinate

$0 = \frac{-2.n + 4.m}{m+n} \Rightarrow m : n$ is 1 : 2.

ii) Distance between (3, 4) and (– 1, 7) is $\sqrt{4^2+3^2} = 5$.

Distance between $(a,3)$ and $(-1,7)$

is $\sqrt{(a+1)^2+(3-7)^2} = \sqrt{(a+1)^2+4^2} = 5$.

$\Rightarrow a + 1 = 3 \Rightarrow a = 2$. The point is (2, 3).

iii) Let the points are not collinear. Then they will be vertices of a triangle. Area of that triangle is $\frac{1}{2}\{-2(1+1)+(-6)(-1-3)+(-10)(3-1)\} = 0$. It is a contradiction. So points are collinear.

iv) $r = 2a\cos\theta. \Rightarrow r^2 = 2ar\cos\theta. \Rightarrow x^2 + y^2 = 2ax.$

v) The straight line parallel to the straight line $2x - 3y + 8 = 0$ is $2x - 3y + k = 0$. Since it passes through (2, 3) we get $2\times2 - 3\times3 + k = 0 \Rightarrow k = 5$. The equation is $2x - 3y + 5 = 0$

Ex 3. [2010]

i) Write down any equation parallel to the X-axis and slope of the X-axis.

ii) Show that the points (4, 4), (6, 2) and (7, 1) are collinear.

iii) Find the equation of the straight line parallel to $x=2y$ and passing through (1, 1).

iv) In what ratio the line joining the points (2, 3) and (– 6, 8) will be divided by Y-axis?

v) Find the equation of the perpendicular bisector of the line joining the points A(– 4, 10) and B(4, – 10).

12. Prove that the following lines are concurrent-

i) $2x+6y+1=0,\ 6x-3y-4=0,\ 12x+21y+1=0$

ii) $y-x+1=0,\ 2y-x-1=0,\ 7x-y-19=0$

iii) $4x-y+2=0,\ x-3y+3=0,\ 3x+2y=1$

13. Find angle between the lines:

i) $x+2y-1=0$ and $6x+5y-3=0$

ii) $3x-2y-4=0$ and $2x+7y+3=0$

iii) $x+y-2=0$ and $2x+3y-3=0$

iv) $x+3y-1=0$ and $2x+6y+5=0$

v) $5x-4y+2=0$ and $4x+5y+3=0$

14. Find equation to the line passing through point of intersection of

i) $2x-3y+5=0$ and $x+2y-3=0$ and through the point $(3, 4)$

ii) $x-7y+2=0$ and $3x-5y-3=0$ and parallel to $x-y-2=0$

iii) $x-2y-1=0$ and $2x-7y-3=0$ and perpendicular to $3x+y+5=0$

15. Show that the following lines are concurrent and find their common point:

i) $2x-y-1=0,\ 7x-8y+1=0$ and $11x-5y-6=0$

ii) $3x-2y=0,\ x+2y-8=0$ and $5x-3y-1=0$

iii) $2x-7y+10=0,\ x-7y+12=0$ and $3x-2y-2=0$

16. Find distance between the parallel lines:

i) $3x-3y+1=0$ and $x-y+5=0$.

ii) $5x+4y-5=0$ and $10x+8y+3=0$.

17. Show that following points are collinear:

i) $(2, 3)$, $(3, 5)$ and $(6, 11)$.

ii) $(-1, 2)$, $(3, 1)$ and $(7, 0)$.

18. Find reflection of

i) $(2, 0)$ on Y-axis.

ii) $(-3, 7)$ on Y-axis.

iii) $(3, 4)$ on $3x+y-18=0$

Answer

1. $x^2+y^2-8x-2y+8=0$ 2. $x^2+y^2=36$

3. $7x+5y+16=0$ 4. $8(x^2+y^2)-2x-36y+35=0$

5. $y^2=4a(x-a)$

6. i) $y+x-1=0$ ii) $y+x-5=0$

7. i) $4y+7x-36=0$ ii) $y+3x-8=0$

8. i) $y=3x+4$ ii) $y+3x+2=0$

iii) $3x-4y+7=0$ iv) $y=x+2$

9. i) $y=\frac{3}{4}x+\frac{5}{8}$ ii) $\frac{x}{-\frac{5}{6}}+\frac{y}{\frac{5}{8}}=1$ iii) $-\frac{3}{5}x+\frac{4}{5}y=\frac{1}{2}$

10. i) $\frac{8}{\sqrt{170}}$ ii) $\frac{9}{\sqrt{26}}$ iii) $\frac{14}{5}$

11. i) $(\frac{1}{12},\frac{1}{4})$ ii) $(-3, 3)$ iii) $(-2, 0)$

13. i) $\tan^{-1}\frac{7}{16}$ ii) $\tan^{-1}\frac{25}{8}$ iii) $\tan^{-1}\frac{1}{5}$

iv) parallel lines v) perpendicular lines.

14. i) $17x-22y+37=0$ ii) $8x-8y-11=0$ iii) $3x-9y-4=0$

15. i) $(1, 1)$ ii) $(2, 3)$ iii) $(2, 2)$

16. i) $\frac{14}{3\sqrt{2}}$ ii) $\frac{13}{2\sqrt{41}}$

18. i) $(-2, 0)$ ii) $(3, 7)$ iii) $(6, 5)$

Distance of $6x+2y+5=0$ from origin is

$$-\frac{5}{\sqrt{6^2+2^2}}=-\frac{5}{2\sqrt{10}}$$

Distance between the two lines is $\frac{3}{\sqrt{10}}-(-\frac{5}{2\sqrt{10}})=\frac{11}{2\sqrt{10}}$

Alternately $(1,0)$ is a point on $3x+y-3=0$.

Distance of $6x+2y+5=0$ from $(1,0)$ is

$$-\frac{6\times1+2\times0+5}{2\sqrt{10}}=-\frac{11}{2\sqrt{10}} \text{ i. e. } \frac{11}{2\sqrt{10}}$$

ii) $(5,0)$ is a point on $x+4y-5=0$

Distance of $3x+12y-8=0$ from $(5,0)$ is

$$-\frac{3\times5+12\times0-8}{\sqrt{3^2+12^2}}=-\frac{7}{3\sqrt{17}} \text{ i. e. } \frac{7}{3\sqrt{17}}$$

Ex 13. Find reflection of $(1,2)$ on $y=x$

Solution: Let reflection of $(1,2)$ on $y=x$ is (a,b)

Distance of $(1,2)$ from $y=x$ i.e. $x-y=0$ is $-\frac{1\times1-1\times2}{\sqrt{2}}=\frac{1}{\sqrt{2}}$

Distance of (a,b) from $y=x$ i.e. $x-y=0$ is $-\frac{1\times a-1\times b}{\sqrt{2}}=-\frac{1}{\sqrt{2}}$

$\Rightarrow a-b=1$ ....(1)

Gradient of the line joining $(1,2)$ and

(a,b) is $\frac{b-2}{a-1}=-1$ or, $a+b=3$ ....(2)

Solving (1) and (2) we get $a=2$ and $b=1$.

Hence reflected point is $(2,1)$.

Exercise 4.2

1. Find locus of a point moving at a constant distance 3 from $(4,1)$.
2. Find locus of a point moving at a constant distance 6 from $(0,0)$.
3. Find locus of the point moving equidistant from $(-4,-5)$ and $(3,0)$.
4. Find locus of the moving point P if it moves such that its distance from $(-1,0)$ is three times its distance from $(0,2)$.
5. Find locus of the moving point P if it moves such that its distance from $(2a,0)$ is equal to its distance from Y-axis.
6. Find equation to the line
   i) which makes angle $-45°$ with X-axis and has y-intercept 1.
   ii) having gradient -1 and passing through $(4,1)$.
7. Find equation to the line passing through
   i) $(4,2)$ and $(0,9)$ ii) $(2,2)$ and $(3,-1)$.
8. Find equation to the line passing through $(-1,1)$ and
   i) having gradient 3
   ii) having y-intercept -2
   iii) passing through $(3,4)$
   iv) cuts off equal distances from two axes.
9. The equation of a straight line is $6x-8y+5=0$. Write the i) gradient form, ii) intercept form and iii) perpendicular form.
10. Find length of perpendicular from
    i) $(2,1)$ to $11x-7y-7=0$
    ii) $(0,0)$ to $x-5y-9=0$
    iii) $(-1,1)$ to $4x-3y-7=0$
11. Find point of intersection of the pair of lines:
    i) $6x+2y-1=0$ and $3x-5y+1=0$
    ii) $x+y=0$ and $2x-y+9=0$
    iii) $x+4y+2=0$ and $2x-y+4=0$

x-intercept is $\frac{1}{3}$ and y-intercept is $\frac{1}{4}$

iii) Perpendicular form is $\frac{3}{\sqrt{3^2+4^2}}x+\frac{4}{\sqrt{3^2+4^2}}y=\frac{1}{\sqrt{3^2+4^2}}$

$$\Rightarrow \frac{3}{5}x+\frac{4}{5}y=\frac{1}{5}$$

Length of perpendicular to the line from origin is $\frac{1}{5}$.

Length of perpendicular to the line from $(2, 3)$ is

$$-\frac{3\times 2+4\times 3-1}{5}=-\frac{17}{5} \text{ i. e. } \frac{17}{5}$$

Length of perpendicular to the line from $(-4, 1)$ is $-\frac{3\times(-4)+4\times 1-1}{5}=\frac{9}{5}$

We see origin and $(-4, 1)$ lie on same side of the line whereas $(2, 3)$ lies on other side.

Ex 10. Find point of intersection of the two lines $4x+7y-6=0$ and $5x+11y-3=0$. What is the angle between them? Find equation to the line passing through the point of intersection and $(2, 1)$.

Solution: Solution of the two equations $4x+7y-6=0$ and $5x+11y-3=0$ for x and y are $(5, -2)$. So point of intersection is $(5, -2)$.

Gradient of $4x+7y-6=0$ is $m_1=-\frac{4}{7}$

and gradient of $5x+11y-3=0$ is $m_2=-\frac{5}{11}$.

Angle between them is $\tan^{-1}\frac{m_2-m_1}{1+m_1m_2}=\tan^{-1}\frac{-\frac{5}{11}+\frac{4}{7}}{1+\frac{5}{11}\times\frac{4}{7}}=\tan^{-1}\frac{9}{97}$

Equation to the line passing through the point of intersection is $4x+7y-6+k(5x+11y-3)=0$ where k has any value.
When it passes through $(2, 1)$

we get $4.2+7.1-6+k(5.2+11.1-3)=0 \Rightarrow k=-\frac{1}{2}$

Hence equation is $x+y-3=0$

Ex 11. Find equation to the line passing through

i) $(1, 2)$ and parallel to $6x+2y-3=0$

ii) $(-3, 1)$ and perpendicular to $2x+9y-1=0$

iii) $(4, -4)$ and centroid of the triangle with vertices $(1, 1), (3, 2)$ and $(5, -3)$

Solution: i) Equation to the line parallel to $6x+2y-3=0$ is $6x+2y+k=0$
$(1, 2)$ is a point on the line. So, $6\times 1+2\times 2+k=0$.
Hence $k=-10$.
The required equation is $6x+2y-10=0$

ii) Equation to the line perpendicular to $2x+9y-1=0$ is $9x-2y+k=0$. $(-3, 1)$ is a point on the line. So, $9\times(-3)-2\times 1+k=0$.
Hence $k=29$
The required equation is $9x-2y+29=0$

iii) Centroid of the triangle with vertices $(1, 1), (3, 2)$ and $(5, -3)$ is

$(\frac{1+3+5}{3}, \frac{1+2-3}{3})$ i.e. $(3, 0)$.

Equation to the line passing through $(4, -4)$ and $(3, 0)$ is

$$\frac{-4-0}{4-3}=\frac{y-0}{x-3}\Rightarrow 4x+y=12$$

Ex 12. Find distance between the parallel lines:

i) $3x+y-3=0$ and $6x+2y+5=0$.

ii) $x+4y-5=0$ and $3x+12y-8=0$.

Solution: i) Distance of $3x+y-3=0$ from origin is $-\frac{-3}{\sqrt{3^2+1^2}}=\frac{3}{\sqrt{10}}$

xi) The general equation to the straight line passing through the point of intersection of
$x+2y+5=0$ and $x-y+7=0$ is $x+2y+5+k(x-y+7)=0$.

Gradient of the line is $m=\frac{k+1}{k-2}$. Since the line is parallel to $5x-2y+1=0$ so $\frac{k+1}{k-2}=\frac{5}{2}$

$\Rightarrow$ $k=4$. *Hence the required equation to the straight line is*
$x+2y+5+4(x-y+7)=0$.
or, $5x-2y+33=0$.

Ex 8. [2013] A ray of light passes through a point (8, 3) and is reflected at (14, 0) on the x-axis. Find the equation of the reflected ray.

ii) Express $\sqrt{3}x+y=8$ in the perpendicular form, also find the distance of the line from origin.

iii) [2016] Show that the lines $2x-y+8=0$, $3x+y+2=0$ and $4x+3y-4=0$ are concurrent.

iv) The points (11, 2), (3, – 4), (– 7, – 9) are the three vertices of a rectangle. Find the coordinates of the fourth vertex.

v) Find equation of the line which is parallel to the line $5x+4y+7=0$ and passes through the point (2, – 3).

Solution:

i) The reflected ray passes through (14, 0) and (20, 3). Gradient of the line is $\frac{3-0}{20-14}=\frac{1}{2}$ $\Rightarrow$ Equation of the reflected ray is $\frac{y-0}{x-14}=\frac{1}{2}$ $\Rightarrow x-2y=14$

ii) Perpendicular form of the line is $\frac{\sqrt{3}x}{\sqrt{(\sqrt{3})^2+1^2}}+\frac{y}{\sqrt{(\sqrt{3})^2+1^2}}=\frac{8}{\sqrt{(\sqrt{3})^2+1^2}}$

or, $\frac{\sqrt{3}}{2}x+\frac{1}{2}y=4$ Distance of the line from origin is 4

iii) For this we have to show that $\begin{vmatrix}2 & -1 & 8\\ 3 & 1 & 2\\ 4 & 3 & -4\end{vmatrix}=0$

And we see $2(-4-6)+1(-12-8)+8(9-4)=0$ Hence the three lines are concurrent.

iv) Vertices are A(11, 2), B(3, – 4), C(– 7, – 9) and D(x, y).

$AB=\sqrt{(11-3)^2+(2+4)^2}=10$, $BC=\sqrt{(3+7)^2+(-4+9)^2}=\sqrt{125}$

$CD=\sqrt{(-7-x)^2+(-9-y)^2}=\sqrt{x^2+y^2+14x+18y+130}$

$DA=\sqrt{(x-11)^2+(y-2)^2}=\sqrt{x^2+y^2-22x-4y+125}$

We make $AB=CD \Rightarrow x^2+y^2+14x+18y+130=100$

$BC=DA \Rightarrow x^2+y^2-22x-4y+125=125$

$\Rightarrow 36x+22y+30=0 \Rightarrow 18x+11y+15=0$

Gradient of AB is $\frac{2+4}{11-3}=\frac{3}{4}$, Gradient of CD is $\frac{y+9}{x+7}=\frac{3}{4} \Rightarrow 3x-4y-15=0$

$\Rightarrow$ $18x-24y-90=0$ $\Rightarrow 35y+105=0$ $\Rightarrow y=-3,\ x=1$

Hence D is D(1, – 3)

v) General equation to the line parallel to the line $5x+4y+7=0$ is $5x+4y+k=0$.
If (2, – 3) lies on $5x+4y+k=0$ we have $5\times 2+4(-3)+k=0 \Rightarrow k=2$
Required equation of the line is $5x+4y+2=0$

Ex 9. [2014]

i) Check whether the lines $x+3y+5=0$ and $6x-2y+7=0$ are at right angles.

ii) Find equation of the line which passes through the point (4, – 5) and is parallel to the line $3x+4y+5=0$.

iii) Prove that the three points (–2, –2), (2, 2) and $(-2\sqrt{3}, 2\sqrt{3})$ are the vertices of an equilateral triangle.

iv) Find the ratio in which the point (– 11, 16) divides the line segment joining the points (– 1, 2) and (4, – 5).

v) Find the polar coordinates of the point whose Cartesian coordinates are $(1, \sqrt{3})$.

ii) Find locus of the point equidistant from $(-4, -5)$ and $(3, 0)$.

iii) [2017] Find intercepts on axes by the straight line $2x + 3y - 5 = 0$.

iv) Find equation to the straight line passing through (0,0) and perpendicular to $x - 6y + 1 = 0$.

v) Find the distance between the two lines $4x + 2y = 1$ and $4x + 2y = 7$.

vi) Find the point of intersection of the two lines $2x + y = 5$ and $x + 2y = 7$.

vii) Find the locus of the set of points equidistant from $A(a+b,\ b-a)$ and $B(a-b,\ a+b)$.

viii) [2016] Change $r = a\sin 2\theta$ to cartesian form.

ix) [2016] y-axis divides the line joining A(3, 4) and B(5, - 7). Find ratio of division.

x) Find the area of the triangle whose vertices are (- 3, - 4), (- 2, 3) and (1, 2).

xi) Find the equation of the straight line passing through the point of intersection of $x + 2y + 5 = 0$ and $y = x + 7$ and parallel to the line $5x - 2y + 1 = 0$.

Solution:

i) Let us find the length of line segments AB, BC, CD, DA, AC and BD.

$AB^2 = (4-1)^2 + (1-5)^2 = 25$, $BC^2 = (1+2)^2 + (5-1)^2 = 25$

$CD^2 = (-2-1)^2 + (1+3)^2 = 25$, $DA^2 = (1-4)^2 + (-3-1)^2 = 25$

$AC^2 = (4+2)^2 + (1-1)^2 = 36$, $BD^2 = (1-1)^2 + (5+3)^2 = 64$

We see the sides of the quadrilateral ABCD are equal and the diagonals are unequal. So ABCD is a rhombus.

ii) Let the moving point be P(x, y). Square of distance of P from A(–4,–5) is $AP^2 = (x+4)^2 + (y+5)^2$

Square of distance of P from B(3, 0) is $BP^2 = (x-3)^2 + (y-0)^2$

As $AP = BP$, so $AP^2 = BP^2$

$\Rightarrow x^2 + 8x + 16 + y^2 + 10y + 25 = x^2 - 6x + 9 + y^2$

$\Rightarrow 14x + 10y + 32 = 0$

$\Rightarrow 7x + 5y + 16 = 0$.

So locus of P is the straight line given by $7x + 5y + 16 = 0$.

iii) $2x + 3y - 5 = 0$ is equivalent to $\frac{x}{5/2} + \frac{y}{5/3} = 1$

Hence intercept on x-axis is $\frac{5}{2}$ and intercept on y-axis is $\frac{5}{3}$.

iv) The general equation to the straight line perpendicular to $x-6y+1=0$ is $6x+y+K=0$.

Since (0, 0) is a point on the line so K = 0. So equation to the straight line passing through (0, 0) and perpendicular to $x - 6y + 1 = 0$ is $6x + y = 0$.

v) Distance of $4x + 2y = 1$ from (0, 0) is $\frac{1}{\sqrt{4^2+2^2}} = \frac{1}{\sqrt{20}}$

vi) The two lines are $2x + y = 5$ and $x + 2y = 7$

We solve it for x and y. $x + 2y = 7$ or, $2x + 4y = 14$

So $(2x + 4y) - (2x + y) = 14 - 5 = 9$ or, $y = 3$. This gives $x = 1$.

Hence point of intersection of the two lines is (1, 3).

vii) Let $P(x, y)$ be the moving point.

$AP^2 = (a+b-x)^2 + (b-a-y)^2 \quad BP^2 = (a-b-x)^2 + (a+b-y)^2$

$AP^2 = BP^2$ So $(a+b-x)^2 + (b-a-y)^2 = (a-b-x)^2 + (a+b-y)^2$

$\Rightarrow (a+b)^2 + (b-a)^2 + x^2 + y^2 - 2(a+b)x - 2(b-a)y$

$= (a-b)^2 + (a+b)^2 + x^2 + y^2 - 2(a-b)x - 2(a+b)y$

$\Rightarrow ax + bx + by - ay = ax - bx + ay + by \Rightarrow bx = ay$

viii) $r = a\sin 2\theta \Rightarrow r^3 = 2ar^2 \sin\theta\cos\theta = 2axy \Rightarrow r^6 = (2axy)^2 \Rightarrow (x^2+y^2)^3 = (2axy)^2$

ix) $R(0, a)$ is a point on y-axis which divides AB in the ratio $m : n$.

$\Rightarrow \frac{5m+3n}{m+n} = 0 \Rightarrow m : n = -3 : 5$. It is external division.

x) Area $= \frac{1}{2}\{(-3\times 3) - (-4\times -2) + (-2\times 2) - (3\times 1) + (1\times -4) - (2\times -3)\}$

$= \frac{1}{2}(-9-8-4-3-4+6) = -11$ Area is 11 unit.

iii) Let the point be (x, y). Its distance from $(-1, 0)$ is $\sqrt{(x+1)^2+(y-0)^2}$

Its distance from $(0, 2)$ is $\sqrt{(x-0)^2+(y-2)^2}$

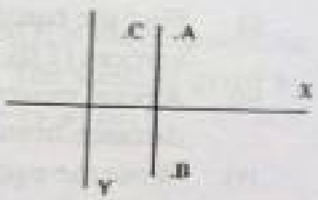
Given $\sqrt{(x+1)^2+(y-0)^2} = 3\sqrt{(x-0)^2+(y-2)^2}$

$\Rightarrow x^2+y^2+2x+1=9(x^2+y^2-4y+4)$

$\Rightarrow 8x^2+8y^2-2x-36y+35=0$

iv) Given line is $y=6x-1$. or, $6x-y=1$. Intercept form is $\dfrac{x}{\frac{1}{6}}+\dfrac{y}{-1}=1$ x-intercept is $\frac{1}{6}$, y-intercept is -1.

v) The perpendicular distance is $\dfrac{6.1-5.1+3}{\sqrt{6^2+5^2}}=\dfrac{4}{\sqrt{61}}$



vi) Reflection of $A(3, 2)$ on X-axis is $B(3, -2)$
Reflection of $A(3, 2)$ on $x=2$ is $C(1, 2)$
Distance of A from X-axis is 2, distance of B from X-axis is 2 and lie opposite to A.
Distance of A from $x=2$ is 1, distance of C from $x=2$ is 1 and lie opposite to A.

vii) Equation to the line perpendicular to $2x-6y+1=0$ is $6x+2y+k=0$. Since it passes through $(1, 2)$ so $6.1+2.2+k=0$. Hence $k=-8$ and equation is $6x+2y-8=0$.

Ex 6. [2011]

i) **[2016]** Find the area of the quadrilateral formed by the points $(0, 3)$, $(2, -1)$, $(5, 0)$ and $(4, 6)$ taken in order.

ii) Show that the points $(-2, 4)$, $(1, 1)$ and $(3, -1)$ are collinear.

iii) Find the angle made by the line joining $(-3, -5)$ and $(5, 2)$ with the X-axis.

iv) Express $\frac{x}{2}+\frac{y}{3}=1$ in perpendicular form.

v) **[2010, 2016]** Find the equation of the straight line which passes through the point $(2, 3)$ such that the sum of the intercepts on the axes is 10.

vi) Find the equation of the straight line passing through $(5, 4)$ and the point of intersection of $2x+3y-1=0$ and $3x-4y+7=0$.

Solution:

i) Let the points be $A(0, 3)$, $B(2, -1)$, $C(5, 0)$ and $D(4, 6)$.
Area of ABCD = Area of triangle ABC + Area triangle ADC
$=\frac{1}{2}\{0(-1-0)+2(0-3)+5(3+1)\}+\{0(0-6)+5(6-3)+4(3-0)\}=\frac{41}{2}$

ii) Let the points $A(-2, 4)$, $B(1, 1)$ and $C(3, -1)$ are not collinear. Then ABC is a triangle. Its area is $\frac{1}{2}\{-2(1+1)+1(-1-4)+3(4-1)\}=0$. A contradiction. Hence, they are collinear.

iii) The gradient of the line joining $(-3, -5)$ and $(5, 2)$ is $\frac{2+5}{5+3}=\frac{7}{8}$

The angle made by the line with X-axis is $\tan^{-1}\frac{7}{8}$

iv) $\frac{x}{2}+\frac{y}{3}=1$ or, $3x+2y=6$. The perpendicular form is

$$\frac{3x}{\sqrt{3^2+2^2}}+\frac{2y}{\sqrt{3^2+2^2}}=\frac{6}{\sqrt{3^2+2^2}} \text{ or, } \frac{3x}{\sqrt{13}}+\frac{2y}{\sqrt{13}}=\frac{6}{\sqrt{13}}$$

v) Let the straight line is $\frac{x}{a}+\frac{y}{10-a}=1$. $(2, 3)$ is a point on the line

$\Rightarrow \frac{2}{a}+\frac{3}{10-a}=1 \Rightarrow 20-2a+3a=10a-a^2 \Rightarrow a^2-9a+20=0 \Rightarrow a=4, 5$

The equation is $\frac{x}{4}+\frac{y}{6}=1$ and $\frac{x}{5}+\frac{y}{5}=1$ or $x+y=5$.

vi) The straight line through the point of intersection of $2x+3y-1=0$ and $3x-4y+7=0$ is $2x+3y-1+k(3x-4y+7)=0$. $(5, 4)$ is a point on the line means $2.5+3.4-1+k(3.5-4.4+7)=0 \Rightarrow k=-\frac{7}{2} \Rightarrow$ the equation is $17x-34y+51=0$

Ex 7. [2012]

i) Show that the points $A(4, 1)$, $B(1, 5)$, $C(-2, 1)$ and $D(1, -3)$ are the vertices of a rhombus.

Solution:

i) One equation to the straight line parallel to the X-axis is $y = 4$. Slope of the X-axis is 0.

ii) The equation to the line passing through (4, 4) and (6, 2) is
$\frac{y-4}{x-4} = \frac{2-4}{6-4} = -1$ i.e. $x + y = 8$.
The equation to the line passing through (4, 4) and (7, 1) is
$\frac{y-4}{x-4} = \frac{7-4}{1-4} = -1$ i.e. $x + y = 8$. Thus, they are collinear.

iii) The equation of the straight line parallel to $x = 2y$ is $x = 2y + k$. (1, 1) is a point on the line implies $1 = 2.1 + k$. Means $k = -1$. Hence the equation to the line is $x - 2y + 1 = 0$.

iv) The x-coordinate of the point is 0. Let the ratio be $m : n$ and point is $(0, p)$.
$0 = \frac{2n - 6m}{m + n}$. $m : n$ is $1 : 3$.

v) The locus of the point moving equidistant from A and B is the perpendicular bisector of AB. Let the moving point be $P(x, y)$. The locus is given by PA = PB $\Rightarrow (x + 4)^2 + (y - 10)^2 = (x - 4)^2 + (y + 10)^2$
$\Rightarrow 16x = 40y \Rightarrow 2x = 5y$

Ex 4. [2010]

i) Find the coordinates of the centroid of the triangle with vertices at (1, 2), (3, 4) and (–1, 3).

ii) The polar equation of a curve is given by $r = 2a \sin\theta$, Find its Cartesian equation.

iii) If points (a, b), (c, d), $(a - c, b - d)$ lie on a straight line, prove that $ad = bc$.

iv) Find the coordinates of the point which divides the line segment joining the points (2, 3) and (4, –5) internally in the ratio 2:3.

Solution:

i) Let (p, q) be the centroid. Then $p = \frac{1+3-1}{3} = 1$ and $q = \frac{2+4+3}{3} = 3$. So centroid is (1, 3).

ii) $r = 2a \sin\theta$. $\Rightarrow r^2 = 2ar \sin\theta$. $\Rightarrow x^2 + y^2 = 2ay$.

iii) If the given points are on a straight line then gradients of the lines joining any two points are same. Gradient of the line joining (a, b), (c, d) is $\frac{d-b}{c-a}$ and that of the line joining (c, d), $(a - c, b - d)$ is $\frac{b-2d}{a-2c}$.

Hence $\frac{d-b}{c-a} = \frac{b-2d}{a-2c}$
$\Rightarrow ad - ab - 2cd + 2bc = bc - 2cd - ab + 2ad \Rightarrow ad = bc$.

iv) Let (p, q) be the point. Then $p = \frac{2\times3+4\times2}{2+3} = \frac{14}{5}$ and q
$= \frac{3\times3-2\times5}{2+3} = -\frac{1}{5}$ So point is $(\frac{14}{5}, -\frac{1}{5})$.

Ex 5. [2011]

i) Convert the polar coordinates (8, –300°) to Cartesian form.

ii) Find the equation to the median from the vertex (1, 4) of the triangle with vertices (2,0), (1, 4) and (3, 3).

iii) A point moves so that its distance from the point (–1, 0) is always 3 times its distance from the point (0, 2). Find the equation to the locus.

iv) Write the equation $y = 6x - 1$ in intercept form. What are the intercepts on the axes?

v) Find length of perpendicular from (1, 1) to the straight line $6x - 5y + 3 = 0$.

vi) Find reflection of (3, 2) on X-axis. What is its reflection on $x = 2$?

vii) Find equation to the straight line passing through (1, 2) and perpendicular to $2x - 6y + 1 = 0$.

Solution :

i) $x = r \cos\theta = 8 \cos(-300°) = 8 \cos 60° = 4$.
$y = r \sin\theta = 8 \sin(-300°) = 8 \sin 60° = 8 \times \frac{\sqrt{3}}{2} = 4\sqrt{3}$ The point is $(4, 4\sqrt{3})$.

ii) The median joins the points (1, 4) and $(\frac{2+3}{2}, \frac{0+3}{2})$. Equation is $\frac{y-4}{x-1}$
$= \frac{4-1.5}{1-2.5} = -\frac{5}{3}$ or, $5x + 3y - 17 = 0$.

xi) It is the locus of the point moving at equal distances from A and B

If the moving point is (x, y), the locus is

$$\sqrt{(x+4)^2+(y-10)^2}=\sqrt{(x-4)^2+(y+10)^2}$$

$\Rightarrow (x+4)^2-(x-4)^2=(y+10)^2-(y-10)^2 \qquad \Rightarrow 2x = 5y.$

xii) General equation to the line passing through the point of intersection of the lines $x+y-2=0$ and $2x-3y+1=0$ is

$x+y-2+l(2x-3y+1)=0$

$(1,-1)$ lies on the line $\Rightarrow 1+(-1)-2+l\{2\times 1-3(-1)+1\}=0$

$\Rightarrow l=\frac{1}{3} \Rightarrow$ Equation to the straight line is $x+y-2+\frac{1}{3}(2x-3y+1)=0 \Rightarrow x=1$

xiii) Let the moving point be (x, y). Given $\sqrt{(x-h)^2+(y-k)^2}=k$

$\Rightarrow x^2+y^2-2hx-2ky+h^2=0.$

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Ex 10. [2015]

i) If the distance between $(2, x)$ and $(1, 9)$ is $5\sqrt{2}$, find x.

ii) Write down the equation of a line parallel to x-axis at a distance of 5 units from x-axis.

iii) Reduce $3x+4y-12=0$ to intercept form.

iv) Find the condition that the point (a, b) should lie on the line joining $(2, 3)$ and $(-4, -1)$.

v) Find the coordinates of the point which divides the line joining the points $(7, 8)$ and $(-6, 11)$ internally in the ratio $5:7$.

vi) If the sum of squares of the distance of a variable point from the two points $(-1, 2)$ and $(3, 4)$ are constant, find the locus of the point.

vii) Find the equation of the line passing through $(3, 5)$ and perpendicular to the line joining $(4, 2)$ and $(2, 8)$.

viii) If the point (a, b), (b, a) and (x, y) are collinear, prove that $x+y=a+b$

ix) In what ratio is the line joining the points $(2, 3)$ and $(-5, 1)$ divided by y-axis?

x) Find the equation to the line which makes an angle 150° with x-axis and cuts 3 units from y-axis.

xi) Find the equation to the perpendicular bisector of the line joining the points $A(-4, 10)$ and $B(4, -10)$.

xii) Find equation to the straight line passing through the point of intersection of the lines $x+y-2=0$ and $2x-3y+1=0$ and the point $(1, -1)$.

xiii) **[2016]** Write the locus of a point equidistant from a fixed point (h, k) where the distance is k.

Solution: i) distance between $(2, x)$ and $(1, 9)$ is

$$\sqrt{(2-1)^2+(x-9)^2}=\sqrt{x^2-18x+82}=5$$

$\Rightarrow x^2-18x+82=50 \Rightarrow x^2-18x+32=0=(x-16)(x-2) \Rightarrow x=16, x=2$

ii) There are two lines parallel to x-axis at a distance of 5 units. They are $y=5$ and $y=-5$.

iii) $3x+4y-12=0 \Rightarrow \frac{x}{4}+\frac{y}{3}=1$

iv) (a, b), $(2, 3)$ and $(-4, -1)$ are collinear if gradient of the line through (a, b) and $(2, 3)$ = gradient of the line through $(-4, -1)$ and $(2, 3)$

$$\Rightarrow \frac{3-b}{2-a}=\frac{3+1}{2+4}=\frac{2}{3} \quad \Rightarrow 2a-3b+5=0$$

v) $R(a, b)$ divides the line joining the points $(7, 8)$ and $(-6, 11)$ internally in the ratio $5:7$ $\Rightarrow a=\frac{5(-6)+7\times 7}{5+7}=\frac{19}{12}$ and $b=\frac{5\times 11+7\times 8}{5+7}=\frac{111}{12}$

Hence R is $R(\frac{19}{12}, \frac{111}{12})$

vi) Let the variable point be (x, y). Given

$\{(x+1)^2+(y-2)^2\}+\{(x-3)^2+(y-4)^2\}=c$

$\Rightarrow 2x^2+2y^2-3x-12y+30=c$

vii) Slope of the line joining $(4, 2)$ and $(2, 8)$ is $\frac{8-2}{2-4}=-3$

Equation of the line passing through $(3, 5)$ having gradient $\frac{1}{3}$ is $\frac{y-5}{x-3}=\frac{1}{3}$

$\Rightarrow x-3y+12=0.$

viii) (a, b), (b, a) and (x, y) are collinear if gradient of the line through (a, b) and (b, a) = gradient of the line through (x, y) and (b, a)

$$\Rightarrow \frac{a-b}{b-a}=\frac{a-y}{b-x}=-1 \quad \Rightarrow a-y=x-b \quad \Rightarrow x+y=a+b$$

ix) Let the ratio be $m:n$. At the point of division x coordinate is 0.

$\Rightarrow \frac{-5m+2n}{m+n}=0 \Rightarrow m:n::2:5$. Hence ratio is $2:5$.

x) $m=\tan 150^\circ=-\frac{1}{\sqrt{3}}$ $\quad c=\pm 3$

The two lines are $y=-\frac{1}{\sqrt{3}}x-3$ and $y=-\frac{1}{\sqrt{3}}x+3$.

vi) Find the distance between two parallel lines $4x + 3y = 8$ and $4x + 3y + 12 = 0$.

vii) Find the equation of the line which passes through (5, 6) and has intercepts on the axes equal in magnitude but opposite in sign.

viii) Show that the lines $x - 3y + 1 = 0$, $2x + y - 4 = 0$, $3y - 8x + 10 = 0$ are concurrent

ix) Find the equation of the bisector of the angle between the lines $4x - 3y + 1 = 0$ and $12x - 5y + 7 = 0$.

Solution: i) m_1 = Gradient of $x + 3y + 5 = 0 = -\frac{1}{3}$

m_1 = Gradient of $6x - 2y + 7 = 0 = 3$

$m_1 \times m_1 = -\frac{1}{3} \times 3 = -1 \Rightarrow$ The two lines are at right angles.

ii) General equation to the line parallel to the line $3x + 4y + 5 = 0$ is $3x + 4y + k = 0$. If (4, – 5) lies on $3x + 4y + k = 0$ we have $3 \times 4 + 4(-5) + k = 0 \Rightarrow k = 8$

Required equation of the line is $3x + 4y + 8 = 0$

iii) Vertices are $A(-2, -2)$, $B(2, 2)$, $C(-2\sqrt{3}, 2\sqrt{3})$

$$AB = \sqrt{(-2-2)^2 + (-2-2)^2} = 4\sqrt{2},$$

$$BC = \sqrt{(2+2\sqrt{3})^2 + (2-2\sqrt{3})^2} = 4\sqrt{2}$$

$$CA = \sqrt{(-2\sqrt{3}+2)^2 + (2\sqrt{3}+2)^2} = 4\sqrt{2}$$

Thus AB = BC = CA the triangle is equilateral.

iv) R(– 11, 16) divides the line segment joining A(– 1, 2) and B(4, – 5)

in the ratio $m : n$. $\Rightarrow -11 = \frac{4m-n}{m+n} \Rightarrow -11m - 11n = 4m - n \Rightarrow -3m = 2n$

$$\Rightarrow 16 = \frac{-5m+2n}{m+n} \Rightarrow 16m + 16n = -5m + 2n \Rightarrow 21m = -14n \Rightarrow -3m = 2n$$

$\Rightarrow \frac{m}{n} = -\frac{2}{3}$ Hence R(– 11, 16) divides AB externally in the ratio 2 : 3.

v) $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$, $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Hence polar coordinates of the point is $(2, \frac{\pi}{3})$.

vi) (2, 0) is a point on $4x + 3y = 8$.

Distance between two parallel lines $4x + 3y = 8$ and $4x + 3y + 12 = 0$ is same as the distance d of (2, 0) to $4x + 3y + 12 = 0$.

$$d = \frac{4\times2 + 3\times0 + 12}{\sqrt{4^2+3^2}} = \frac{20}{5} = 4$$

vii) The line has intercepts a on x-axis and $-a$ on y-axis.

$\Rightarrow$ Equation to the line is $\frac{x}{a} - \frac{y}{a} = 1$. (5, 6) lies on the line implies $\frac{5}{a} - \frac{6}{a} = 1$

$\Rightarrow a = -1$. $\Rightarrow$ Equation to the line is $x - y + 1 = 0$.

viii) The three lines are concurrent if the coefficient determinant is 0.

$$\text{Coefficient determinant} = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ -8 & 3 & 10 \end{vmatrix} = 1(10+12) + 3(20-32) + 1(6+8) = 0$$

Hence they are concurrent.

ix) There are two bisectors.

$$\text{The equation is } \frac{4x-3y+1}{\sqrt{4^2+3^2}} = \pm\frac{12x-5y+7}{\sqrt{12^2+5^2}}$$

$\Rightarrow 13(4x - 3y + 1) = 5(12x - 5y + 7)$ and $13(4x - 3y + 1) = -5(12x - 5y + 7)$

$\Rightarrow 8x + 14y + 22 = 0$ and $112x - 64y + 48 = 0$

$\Rightarrow 4x + 7y + 11 = 0$ and $7x - 4y + 3 = 0$